MET 2014 Answer Key PDF

Physics

1. According to the principle of dimensional homogenity $[p] = \left[\dfrac{a}{V^2}\right]$

$\Rightarrow \quad [a] = [p][V^2] = [ML^{-1}T^{-2}][L^6] = [ML^5T^{-2}]$

Hence, the units of $a = \text{gm} \times \text{cm}^5 \times \text{s}^{-2}$

$= \text{Dyne} \times \text{cm}^4$

2. Given, $[C^2LR] = \left[C^2L^2 \dfrac{R}{L}\right] = \left[(LC)^2\left(\dfrac{R}{L}\right)\right]$

and we know that frequency of LC circuits $f = \dfrac{1}{2\pi}\dfrac{1}{\sqrt{LC}}$. Here the dimension of LC is equal to $[T^2]$.

$\left[\dfrac{L}{R}\right]$ gives the time constant of L - R circuit, so that the dimension of $\dfrac{L}{R}$ is equal to $[T]$.

Hence the required dimensions

$\left[(LC)^2\left(\dfrac{R}{L}\right)\right] = [T^2]^2[T^{-1}] = [T^3]$.

3. Given, number of particle passing from unit area in unit time $= n$

$= \dfrac{\text{Number of particle}}{A \times t} = \dfrac{[M^0 L^0 T^0]}{[L^2][T]} = [L^{-2}T^{-1}]$

$[n_1] = [n_2] =$ Number of particle in unit volume $= [L^{-3}]$

Now, from the given formula $[D] = \dfrac{[n][x_2 - x_1]}{[n_2 - n_1]}$

$= \dfrac{[L^{-2}T^{-1}][L]}{[L^{-3}]}$

$= [L^2 T^{-1}]$

4. Given that $v = 4t^3 - 2t$

$x = \int v\,dt$, $x = t^4 - t^2 + C$, at $t = 0$, $x = 0$

$\Rightarrow \quad C = 0$

When particle is $2m$ away from the origin, then

$2 = t^4 - t^2 \Rightarrow t^4 - t^2 - 2 = 0$

$\Rightarrow (t^2 - 2)(t^2 + 1) = 0$

$\Rightarrow \quad t = \sqrt{2}$ s

$a = \dfrac{dv}{dt} = \dfrac{d}{dt}(4t^3 - 2t)$

$a = 12t^2 - 2$

For $\quad t = \sqrt{2}$ sec

$a = 12 \times (\sqrt{2})^2 - 2 = 22 \text{ m/s}^2$

5. From given a - t graph acceleration is increasing at constant rate

$\therefore \dfrac{da}{dt} = k$ (constant) $\Rightarrow a = kt$ (by integration)

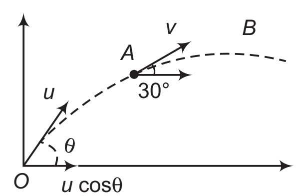
$\Rightarrow \quad \dfrac{dv}{dt} = kt \Rightarrow dv = k\,t\,dt$

$\Rightarrow \quad \int dv = k\int t\,dt \Rightarrow v = \dfrac{kt^2}{2}$

i.e., v is dependent on time parabolically and parabola is symmetric about v-axis and suddenly acceleration becomes zero. *i.e.*, velocity becomes constant. Hence the most stable graph is (c).

6. Let in 2 s body reaches upto point A and after one more sec upto point B.

Total time of ascent for a body is given 3 s



i.e., $\quad t = \dfrac{u \sin\theta}{g} = 3$

$\therefore \quad u \sin\theta = 10 \times 3$

$u \sin\theta = 30 \quad \ldots$(i)

Horizontal component of velocity remains always constant

$u \cos\theta = v \cos 30° \quad \ldots$(ii)

For vertical upward motion between point O and A

$v \sin 30° = u \sin\theta - g \times 2 \quad$ [Using $v = u - gt$]

$v \sin 30° = 30 - 20$ [As $u \sin\theta = 30$]

$\therefore \quad v = 20$ m/s

Substituting this value v, we get in Eq. (ii)

$u \cos\theta = 20 \cos 30° = 10\sqrt{3}$ ...(iii)

From Eqs. (i) and (iii)

$$u = 20\sqrt{3} \text{ and } \theta = 60°$$

7. In non-uniform circular motion two forces will work on a particle F_c and F_t.

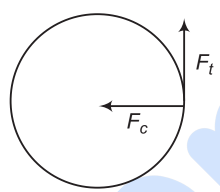
So, the net force $F_{Net} = \sqrt{F_c^2 + F_t^2}$ ...(i)

Centripetal force $F_c = \dfrac{mv^2}{R} = \dfrac{2as^2}{R}$ ...(ii)

[given $\dfrac{1}{2}mv^2 = as^2$ given]

Again from $\dfrac{1}{2}mv^2 = as^2$

$$\Rightarrow \quad v^2 = \frac{2as^2}{m}$$



$$\Rightarrow \quad v = s\sqrt{\frac{2a}{m}}$$

Tangential acceleration $a_t = \dfrac{dv}{dt} = \dfrac{dv}{ds} \cdot \dfrac{ds}{dt}$

$$\Rightarrow \quad a_t = \frac{d}{ds}\left[s\sqrt{\frac{2a}{m}}\right] \cdot v$$

$$a_t = v\sqrt{\frac{2a}{m}} = s\sqrt{\frac{2a}{m}}\sqrt{\frac{2a}{m}} = \frac{2as}{m}$$

and $\quad F_t = ma_t = 2as$ ...(iii)

Now, on substituting value of F_c and F_t in Eq.(i), we get

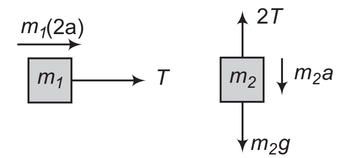
$$\therefore \quad F_{Net} = \sqrt{\left(\frac{2as^2}{R}\right)^2 + (2as)^2} = 2as\left[1 + \frac{s^2}{R^2}\right]^{1/2}$$

8. We have $T_1 = \dfrac{(m_2 + m_3)F}{m_1 + m_2 + m_3}$

Now $T_1 = \dfrac{(3+5)10}{2+3+5} = 8$ N

9. When the block m_2 moves downward with acceleration a, the acceleration of mass m_1 will be $2a$, because it covers double distance in the same time in comparison to m_2.

Let T is the tension in the string.



From the free body diagram of A and B we get

$T = m_1 2a$ ...(i)

$m_2g - 2T = m_2a$ ...(ii)

From Eqs. (i) and (ii), we get

$$a = \frac{m_2g}{(4m_1 + m_2)}$$

10. B will begin to slide on A, if pseudo force is more than limiting friction $F > F_1 \Rightarrow m\left(\dfrac{F}{m+M}\right) > \mu_s R$

$$\Rightarrow \quad m\left(\frac{F}{m+M}\right) > 0.15\,mg$$

$$\therefore \quad F > 1.764 \text{ N}$$

11. We know $F = -\dfrac{dU}{dx}$

$\Rightarrow \quad dU = -F \cdot dx$

$\Rightarrow \quad U = -\int_0^x (-kx + ax^3)\,dx \Rightarrow U = \dfrac{kx^2}{2} - \dfrac{ax^4}{4}$

$\therefore$ We get $U = 0$ at $x = 0$ and $x = \sqrt{\dfrac{2k}{a}}$

Also we get $U =$ negative for $x > \sqrt{\dfrac{2k}{a}}$

From the given function we can see that $F = 0$ at $x = 0$ *i.e.*, slope of U-x graph is zero at $x = 0$.

12. Moment of inertia of rod AB about point $P = \dfrac{1}{12}Ml^2$

MI of rod AB about point $O = \dfrac{Ml^2}{12} + M\left(\dfrac{l}{2}\right)^2 = \dfrac{1}{3}Ml^2$ [from the theorem of parallel axis]

and the system consists of 4 rods of similar type so by the symmetry $I_{system} = \dfrac{4}{3}Ml^2$

13. $\omega_1 = 20 \text{rad/s}, \quad \omega = 0, \quad t = 4\,\text{s}.$ So angular retardation $\alpha = \dfrac{\omega_1 - \omega_2}{t} = \dfrac{20}{4} = 5\,\text{rad/s}^2$

Now, angular speed after 2s $\omega_2 = \omega_1 - \alpha t = 20 - 5 \times 2 = 10$ rad/s

Work done by torque in 2 s = loss in kinetic energy $= \dfrac{1}{2} I(\omega_1{}^2 - \omega_1{}^2) = \dfrac{1}{2}(0.20)((20)^2 - (10)^2)$

$$= \frac{1}{2}(0.20)\,(400 - 100)$$

$$= \frac{1}{2}(0.20) \times (300) = \frac{1}{2} \times 60 = 30$$

14. On the graph stress is represented on x-axis and strain y-axis

So from the graph $y = \cot\theta = \dfrac{1}{\tan\theta} \propto \dfrac{1}{\theta}$

[where, θ is the angle from stress axis]

$\therefore \quad Y_P < Y_Q < Y_R \quad$ [As $\theta_P < \theta_Q < \theta_R$]

We can say that elasticity of wire P is minimum and R is maximum.

15. Energy released

$$= 4\pi TR^3\left(\frac{1}{r} - \frac{1}{R}\right) = 3\left(\frac{4}{3}\pi R^3\right)T\left(\frac{1}{r} - \frac{1}{R}\right)$$

$$= 3VT\left(\frac{1}{r} - \frac{1}{R}\right)$$

16. When substances are mixed in equal volume then density $= \dfrac{\rho_1 + \rho_2}{2} = 4$

$$\Rightarrow \quad \frac{\rho_1 + \rho_2}{2} = 4 \Rightarrow \rho_1 + \rho_2 = 8 \quad \ldots\text{(i)}$$

When substances are mixed in equal masses then density $= \dfrac{2\rho_1\,\rho_2}{\rho_1 + \rho_2} = 3$

$$\Rightarrow \quad 2\rho_1\,\rho_2 = 3(\rho_1 + \rho_2) \quad \ldots\text{(ii)}$$

On solving (i) and (ii) we get

$\rho_1 = 6$ and $\rho_2 = 2$.

17. We have, ideal gas equation $pV = \mu RT = \left(\dfrac{N}{N_A}\right) RT$

where, N = number of molecule, N_A

= Avogadro number

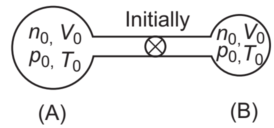
$$\therefore \quad \frac{N_1}{N_2} = \left(\frac{p_1}{p_2}\right)\left(\frac{V_1}{V_2}\right)\left(\frac{T_2}{T_1}\right) = \left(\frac{p}{2p}\right)\left(\frac{V}{V/4}\right)\left(\frac{2T}{T}\right) = \frac{4}{1}$$

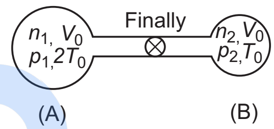
18. Initially for container A, $p_0V_0 = n_0RT_0$

For container B, $p_0V_0 = n_0RT_0$

$$\therefore \quad n_0 = \frac{p_0V_0}{RT_0}$$

Total number of moles $= n_0 + n_0 = 2n_0$





Since, even on heating the total number of moles is conserved

Hence, $\quad n_1 + n_2 = 2n_0 \quad \ldots$(i)

If p be the common pressure, then

For container A, $pV_0 = n_1R\,2T_0 \quad \therefore \; n_1 = \dfrac{pV_0}{2RT_0}$

For container A, $pV_0 = n_2RT_0 \quad \therefore \; n_2 = \dfrac{pV_0}{RT_0}$

Substituting the value of n_0, n_1 and n_2 in Eq. (i) we get

$$\frac{pV_0}{2RT_0} + \frac{pV_0}{RT_0} = \frac{2 \cdot p_0V_0}{RT_0} \Rightarrow p = \frac{4}{3}p_0$$

Number of moles in container A (at temperature $2T_0$)

$$= n_1 = \frac{PV_0}{2RT_0} = \left(\frac{4}{3}p_0\right)\frac{V_0}{2RT_0}$$

$$= \frac{2}{3}\frac{p_0V_0}{RT_0} \qquad \left[\text{As } p = \frac{4}{3}p_0\right]$$

19. At lower pressure, we can assume that given gas behaves as ideal gas so $\dfrac{pV}{RT}$ = constant but when pressure increase, the decrease in volume will not take place in same proportion so $\dfrac{pV}{RT}$ will increases.

20. Since both the liquid and the flask undergoes volume expansion and the flask expands first therefore the level of the liquid initially falls and then rises.

21. $\dfrac{dQ}{dt} = \dfrac{KA\Delta\theta}{l}$, for both rods K, A and $\Delta\theta$ are same

$\therefore \quad \dfrac{dQ}{dt} \propto \dfrac{1}{l}$

so $\dfrac{(dQ/dt)_{\text{semi-circular}}}{(dQ/dt)_{\text{straight}}} = \dfrac{l_{\text{straight}}}{l_{\text{semi-circular}}} = \dfrac{2r}{\pi r} = \dfrac{2}{\pi}$

22. Let two simple harmonic motions are $y = a\sin\omega t$ and $y = a\sin(\omega t + \phi)$

In the first case, $\dfrac{a}{2} = a\sin\omega t \Rightarrow \sin\omega t = \dfrac{1}{2}$

$\therefore \quad \cos\omega t = \dfrac{\sqrt{3}}{2}$

In the second case, $\dfrac{a}{2} = a\sin(\omega t + \phi)$

$\Rightarrow \quad \dfrac{1}{2} = [\sin\omega t \cdot \cos\phi + \cos\omega t \sin\phi]$

$\Rightarrow \quad \dfrac{1}{2} = \left[\dfrac{1}{2}\cos\phi + \dfrac{\sqrt{3}}{2}\sin\phi\right]$

$\Rightarrow \quad 1 - \cos\phi = \sqrt{3}\sin\phi$

$\Rightarrow \quad (1-\cos\phi)^2 = 3\sin^2\phi$

$\Rightarrow \quad (1-\cos\phi)^2 = 3(1-\cos^2\phi)$

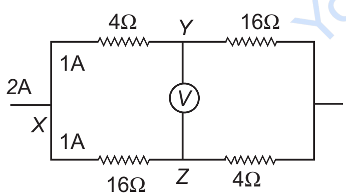
On solving we get

$\cos\phi = +1$ or $\cos\phi = \dfrac{-1}{2}$

i.e., $\quad \phi = 0$ or $\phi = 120°$

23. PD between X and Y is $V_{XY} = V_X - V_Y = 1 \times 4$

and PD between X and Z is $V_{XZ} = V_X - V_Z = 1 \times 16 = 16\text{ V}$



Now, we get potential difference between Y and Z *i.e.*, reading of voltmeter is $V_Y - V_Z = 12\text{ V}$

24. As we know that work done

$$= U_{\text{final}} - U_{\text{initial}} = \frac{1}{2}C(V_2^2 - V_1^2)$$

When potential difference increases from 5V to 10 V then $W = \dfrac{1}{2}C(10^2 - 5^2)$ ...(i)

When potential difference increases form 10 V to 15 V, then $W' = \dfrac{1}{2}C(15^2 - 10^2)$ ...(ii)

On solving Eqs. (i) and (ii) we get

$$W' = 1.67\,W$$

25. $\lambda_{\text{neutron}} \propto \dfrac{1}{\sqrt{T}} \Rightarrow \dfrac{\lambda_1}{\lambda_2} = \sqrt{\dfrac{T_2}{T_1}}$

$\Rightarrow \dfrac{\lambda_1}{\lambda_2} = \sqrt{\dfrac{(273+927)}{(273+27)}} = \sqrt{\dfrac{1200}{300}} = 2 \Rightarrow \lambda_2 = \dfrac{\lambda}{2}$

26. On using $I = \dfrac{P}{A}$; where $P =$ radiation power

$\Rightarrow \quad P = I \times A \Rightarrow \dfrac{nhc}{t\lambda} = IA \Rightarrow \dfrac{n}{t} = \dfrac{IA\lambda}{hc}$

Hence, number of photons entering per sec the eye $\left(\dfrac{n}{t}\right) = \dfrac{10^{-10} \times 10^{-6} \times 5.6 \times 10^{-7}}{6.6 \times 10^{-34} \times 3 \times 10^{8}} = 300$

27. From $E = W_0 + \dfrac{1}{2}mv^2_{\max} \Rightarrow v_{\max} = \sqrt{\dfrac{2E}{m} - \dfrac{2W_0}{m}}$

(where $E = \dfrac{hc}{\lambda}$)

If wavelength of incident light charges from λ to $\dfrac{3\lambda}{4}$ (decreases).

Let energy of incident light charges from E to E' and speed of fastest electron changes from v to v' then

$$v = \sqrt{\frac{2E}{m} - \frac{2W_0}{m}} \quad \text{...(i)}$$

and $\quad v' = \sqrt{\dfrac{2E'}{m} - \dfrac{2W_0}{m}}$ ...(ii)

As $\quad E \propto \dfrac{1}{\lambda} \Rightarrow E' = \dfrac{4}{3}E$

Hence, $\quad v' = \sqrt{\dfrac{2\left(\dfrac{4}{3}E\right)}{m} - \dfrac{2W_0}{m}}$

$\Rightarrow \quad v' = \left(\dfrac{4}{3}\right)^{1/2}\sqrt{\dfrac{2E}{m} - \dfrac{2W_0}{m\left(\dfrac{4}{3}\right)^{1/2}}}$

$\Rightarrow \quad v' = \left(\dfrac{4}{3}\right)^{1/2} X = \sqrt{\dfrac{2E}{m} - \dfrac{2W_0}{m\left(\dfrac{4}{3}\right)^{1/2}}} > v$

So $\quad v' > \left(\dfrac{4}{3}\right)^{1/2} v.$

28. Using $\frac{1}{\lambda} = RZ^2\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right) \Rightarrow \lambda \propto \frac{1}{Z^2} \Rightarrow$

$\lambda_{Li} : \lambda_{He} : \lambda_H = \frac{1}{9} : \frac{1}{4} : \frac{1}{1} = 4 : 9 : 36$

29. By using $N = N_0\left(\frac{1}{2}\right)^{1/T_{1/2}}$

$$\Rightarrow \quad t = \frac{T_{12}\log_e\left(\frac{N_0}{N}\right)}{\log_e(2)}$$

$$\Rightarrow \quad t \propto \log_e\frac{N_0}{N}$$

$$\Rightarrow \quad \frac{t_1}{t_2} = \frac{\log_e\left(\frac{N_0}{N}\right)_1}{\log_e\left(\frac{N_0}{N}\right)_2}$$

Hence, $\frac{6}{10} = \frac{\log_e(8/1)}{\log_e(N_0/N)}$

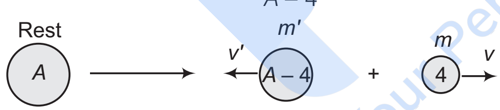
$$\Rightarrow \quad \log_e\frac{N_0}{N} = \frac{10}{6}\log_e(8) = \log_e 32$$

$$\Rightarrow \quad \frac{N_0}{N} = 32$$

So, fraction that decays $= 1 - \frac{1}{32} = \frac{31}{32}$

30. According to conservation of momentum

$4v = (A-4)v' \quad \Rightarrow \quad v' = \frac{4v}{A-4}$



31. ${}_{Z=90}\text{Th}^{A=228} \longrightarrow {}_{Z'=83}\text{Bi}^{A'=212}$

Number of α-particles emitted

$$n_\alpha = \frac{A - A'}{4} = \frac{228 - 212}{4} = 4$$

Number of β-particles emitted

$$n_\beta = 2n_\alpha - Z + Z' = 2 \times 4 - 90 + 83 = 1.$$

32. Radius of each nucleus $R = R_0(A)^{1/3} = 1.2(64)^{1/3}$ $= 4.8\,\text{m}$

Distance between two nuclei $(r) = 2R$

So, potential energy $U = \frac{k \cdot q^2}{r}$

$$= \frac{9 \times 10^9 \times (1.6 \times 10^{-19} \times 29)^2}{2 \times 4.8 \times 10^{-15} \times 1.6 \times 10^{-19}} = 126.15\,\text{MeV}$$

33. At neutral point magnetic field due to magnet = Horizontal component of the earth's magnetic field.

$$\Rightarrow \frac{\mu_0}{4\pi} \cdot \frac{2M}{r^3} = B_H \Rightarrow \frac{10^{-7} \times 2 \times M \times 1}{(0.2)^3} = 0.3 \times 15^4$$

$$\Rightarrow M = 1.2\,\text{amp} \times \text{m}^2 = 1200\,ab - \text{amp} \times \text{cm}^2.$$

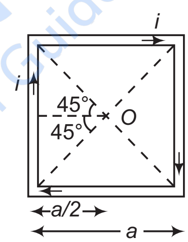
34. By using $T = 2\pi\sqrt{\frac{1}{MB_H}} \Rightarrow \frac{T_1}{T_2} = \sqrt{\frac{(B_H)_2}{(B_H)_1}}$

$$\Rightarrow \quad \frac{60/40}{2.5} = \sqrt{\frac{(B_H)_2}{0.1 \times 10^{-5}}}$$

$$\Rightarrow \quad (B_H)_2 = 0.36 \times 10^{-6}\ \text{T}$$

35. $H = ni \Rightarrow i = \frac{H}{n} = \frac{4 \times 10^3}{500} = 8\ \text{A}$

36. Magnetic field due to one side of the square at centre O.



$$B_1 = \frac{\mu_0}{4\pi} \cdot \frac{2i\sin 45°}{a/2}$$

$$\Rightarrow \quad B_1 = \frac{\mu_0}{4\pi} \cdot \frac{2\sqrt{2}\,i}{a}$$

Hence, magnetic field at centre due to all sides

$$B_{\text{net}} = 4B_1 = \frac{\mu_0(2\sqrt{2}\,i)}{\pi a}$$

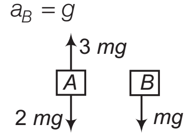
37. $\tau_1 = \frac{8}{9}\mu s$

$$\tau_2 = 18\mu s$$

$$\tau_3 = 4\mu s$$

38. $a_A = \frac{g}{2}$

$$a_B = g$$

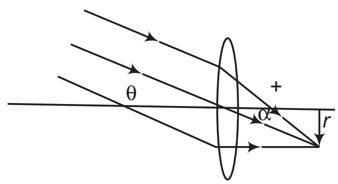


39. $-\frac{1}{F} = \frac{2}{f_l} + \frac{1}{\infty} \Rightarrow F = -\frac{15}{2}$

$$-\frac{2}{15} = \frac{1}{v} - \frac{1}{20}$$

$\Rightarrow v = -12$ cm *i.e.*, 12 cm left of AB.

40. $r = f \tan\alpha$



Hence, $\pi r^2 \propto f^2$

41. Disintegration of each nuclei is independent of any factor. Hence, each nuclei has same change of disintegration.

42. $\frac{1}{f} = \frac{1}{v} - \frac{1}{u} \Rightarrow f = 5\,\text{cm}$

$$f = \frac{uv}{u+v}$$

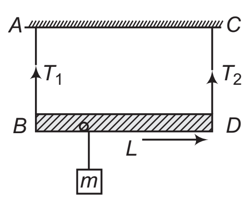
$$\frac{\Delta f}{f} = \left|\frac{\Delta u}{u}\right| + \left|\frac{\Delta v}{v}\right| + \frac{|\Delta u| + |\Delta v|}{|u| + |v|}$$

$$\Delta f = 0.15$$

The most appropriate answer is 5.00 ± 0.10 cm.

43. $\frac{1}{2l}\sqrt{\frac{T_1}{\mu}} = \frac{1}{l}\sqrt{\frac{T_2}{\mu}}$

$$T_2 = \frac{T_1}{4}$$

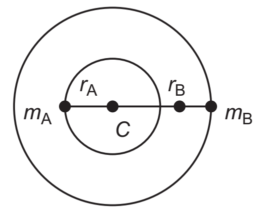


For rotational equilibrium,

$$T_1 x = T_2(L - x)$$

$$\Rightarrow \quad x = \frac{L}{5}$$

44. $\frac{Gm_A m_B}{(r_A + r_B)^2} = \frac{m_A r_A 4\pi^2}{T_A^2} = \frac{m_B r_B 4\pi^2}{T_B^2}$



$$\Rightarrow \quad m_A r_A = m_B r_B$$

$$\therefore \quad T_A = T_B$$

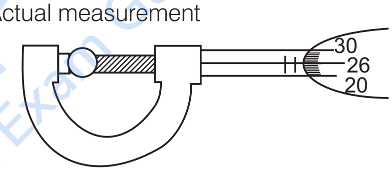
45. $\frac{2}{5}MR^2 = \frac{3}{2}Mr^2$

$$r = \frac{2R}{\sqrt{15}}$$

46. Option (d) is correct.

47. Zero error $= 5 \times \frac{0.5}{50} = 0.05\,\text{mm}$

Actual measurement



$$= 2 \times 0.5\,\text{mm} + 25 \times \frac{0.5}{50} - 0.05\,\text{mm}$$

$$= 1\,\text{mm} + 0.25\,\text{mm} - 0.05\,\text{mm} = 1.20\,\text{mm}$$

48. $\frac{R_1}{R_2} = \frac{A_1}{A_2} = \frac{4}{1}$

$$\frac{P_1}{P_2} = \frac{I^2 R_1}{I^2 R_2} = \frac{4}{1} \Rightarrow \frac{V_1}{V_2} = \frac{IR_1}{IR_2} = \frac{4}{1}$$

$$\frac{J_1}{J_2} = \frac{1}{4}$$

49. $\Delta W = 0 \quad \therefore \quad \Delta Q = \Delta U$

$\Delta Q = \Delta U = I^2 R \Delta t = 1^2(100)(5 \times 60) = 30$ kJ

50. Rate of heat gain $= 1000 - 160 = 840$ J/s

$\therefore$ Required time $= \frac{2 \times 4.2 \times 10^3 \times (77 - 27)}{840}$

$= 500$ s $= 8$ min 20 s

51. $f_1 = \frac{1}{l}\sqrt{\frac{B}{\rho}}$

$$f_2 = \frac{n}{4l}\sqrt{\frac{B}{\rho}} \Rightarrow \frac{f_1}{f_2} = \frac{4}{n} \Rightarrow f_2 = \frac{n}{4}f_1$$

52. Heating of glass bulb is by radiation.

53. Acceleration of the point of suspension

$$a = \frac{d^2y}{dt^2} = 2k = 2\ \text{m/s}^2$$

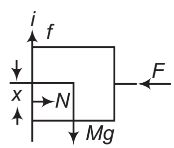
$$T = 2\pi\sqrt{\frac{L}{g_{\text{eff}}}} \Rightarrow T_1 = 2\pi\sqrt{\frac{L}{10}} \text{ and } T_2 = 2\pi\sqrt{\frac{L}{12}}$$

$$\frac{T_1^2}{T_2^2} = \frac{6}{5}$$

54. For equilibrium, $f = Mg$

$$F = N$$

For rotational equilibrium normal will shift downward.

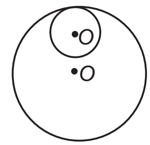


Hence torque due to friction about centre of mass = Torque due to normal reaction about centre of mass.

55. $E_P = \frac{\sigma}{2\varepsilon_0}(-\hat{k}) + \frac{(-2\sigma)}{2\varepsilon_0}(\hat{k}) + \frac{(-\sigma)}{2\varepsilon_0}(\hat{k})$

$$= \frac{-2\sigma}{\varepsilon_0}\hat{k}$$

56. $I_0 = \frac{9MR^2}{2} - \left[\frac{M(R/3)^2}{2} + M\left(\frac{2R}{3}\right)^2\right] = 4MR^2$

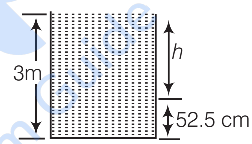


57. Since, B is constant,

$\therefore \qquad \frac{d\phi}{dt} = 0$

$\therefore \qquad i = 0$

58. $u^2 = \frac{2gh}{\left[1 - \left(\frac{A_0}{A}\right)^2\right]} = 50\ (\text{m/s})^2$



59. $B = -\frac{\Delta p}{\left(\frac{\Delta V}{V}\right)} = -\frac{(1.165 - 1.01)\times 10^5}{0.1}$

$$= 1.55 \times 10^5$$

60. $(Z-1)^2\lambda = \text{constant}$

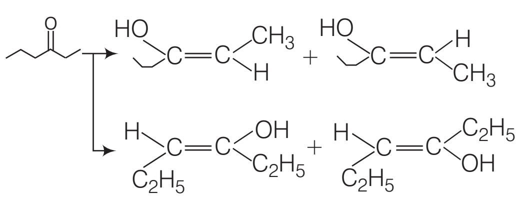
$\therefore \qquad (10^2)\lambda = 4\lambda(Z-1)^2$

$\Rightarrow \qquad Z = 6$

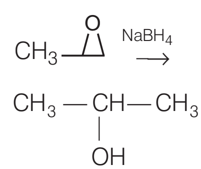
Chemistry

1. Compounds containing acidic hydrogen cannot be used in Grignard's synthesis and the compound CH_2OHCH_2Br contains acidic hydrogen.

2. Four enols, two pairs of geometrical isomers.



3. $CH_3CHO + CH_2N_2 \longrightarrow$



4. As phosphoric acid is a tribasic acid, its n-factor can be 1, 2 or 3. Hence, its normality can be M, $2M$ or $3M$, Where, M is molarity of solution. Therefore, $N = 0.25, 0.50$ or 0.75.

5. $CX_4 + 2O_2 \longrightarrow CO_2 + 2X_2O$

1 mol $\quad$ 2 mol $= 2 \times 32$ g.

9.0 g $\quad$ 1.74 g.

$\because$ 1.74 g O_2 reacts with $CX_4 = 9.0$ g

$\therefore$ 64 g O_2 will react with $CX_4 = \frac{9.0 \times 64}{1.74} = 331$

Molecular mass of $CX_4 = 331$

or $$331 = 12 + 4XX$$

$$X = \frac{331 - 12}{4} = 79.8 \approx 80$$

6. In $S_4O_6^{2-}$, let 'x' be the oxidation number of S.

Thus, the element X may be bromine.

$$4x - 12 = -2$$

$$x = \frac{10}{4} = 2.5$$

7. $$4Au(s) + 8CN^- + O_2(g) + 2H_2O \longrightarrow \underset{\text{soluble complex}}{4[Au(CN)_2]^-} + 4OH^-$$

8. The lanthanide contraction cancels almost exactly the normal size increase on descending a group of transition elements. Thus, Zr and Hf have equal size.

9. Mercurous ion is diamagnetic, this indicates that there is no unpaired electron, hence, mercurous ion is not Hg^+ but Hg_2^{2+}.

10. The high oxidising power of fluorine is due to its low heat of dissociation and high heat of hydration. Due to its smallest size, there is considerable repulsion in the non-bonding electrons which weakens the F—F bond. Thus, bond dissociation energy of F_2 is low.

11. HF has the highest bond dissociation enthalpy amongst all the halogen acids. The hydration enthalpy of the fluoride ion is also highest. The former predominates making HF the weakest of the halogen acids.

12. Pressure, $p = \frac{RT}{V - b} - \frac{a}{V^2}$

$$= \frac{0.082 \times 350}{35 - 0.14} - \frac{20}{(35)^2}$$

$$= 0.807 \text{ atm}$$

13. In a unit cell of KBr, there are four formula units of KBr therefore,

$$\text{Density } (\rho) = \frac{4 \times 119}{6.023 \times 10^{23} \times a^3} = 2.75$$

$$a^3 = 2.873 \times 10^{-22} \text{ cm}^3$$

$$= 2.778 \times 10^{-19} \text{mm}^3$$

Number of unit cells per mm³

$$= \frac{10^{19}}{2.873} = 3.5 \times 10^{18}$$

14. There is one A per unit cell

Number of B per unit cell $= \frac{1}{8} \times 7 = \frac{7}{8}$

Empirical formula $= A_1 B_{7/8}$

$$= A_8 B_7$$

15. State functions do not depend upon path followed.

16. $S_{\text{rhombic}} \longrightarrow S_{\text{monoclinic}}$

$\Delta H = -297948 - (-298246) = 298$ J

$\Delta S = 32.68 - 32.04 = 0.64$

$\Delta G = \Delta H - T\Delta S = 298 - 298 \times 0.64 = 107.28$ J

17. Two angular nodes are found in d-orbitals. Also for a d-orbital to have one radial node it must be $4d$. [$\because$ Number of radial nodes $= n - l - 1$]

18. The valence shell configuration is $4s^2\ 4p^5$ and for the last electron $n = 4, l = 1$

$$m = -1, 0 \text{ or } +1, s = +\frac{1}{2} \text{ or } -\frac{1}{2}$$

Hence, $4, 1, 1, -\frac{1}{2}$

19. The molecular orbital electronic configuration of $CF(15\bar{e})$ is $\sigma 1s^2, \overset{*}{\sigma}1s^2\ \sigma 2s^2\ \overset{*}{\sigma}2s^2\ \sigma 2p_z^2$

$\pi 2p_x^2 \approx \pi 2p_y^2, \overset{*}{\pi}2p_x^1$

Bond order (BO) $= \frac{10 - 5}{2} = 2.5$

In CF^+, there is one electron less in antibonding MO therefore, BO $= \frac{10 - 4}{2} = 3$

In CF^-, there is one electron more in antibonding MO therefore, BO $= \frac{10 - 6}{2} = 2$

Bond length $\propto \frac{1}{\text{Bond order}}$

Hence, the order of bond length in given series is $CF^+ < CF < CF^-$.

20. All are non-polar, molar mass follows the boiling points.

21. In structure II, position of N is changed which is not allowed in resonance.

22. Equilibrium concentration of $N_2O = O_2 = 0.05M$

Let the equilibrium concentration of NO_2 and NO be x M

$$K_c = 0.917 = \frac{(0.05)^2}{x^2} \Rightarrow x = 0.0522 \text{ M}$$

Initial concentrations of NO_2 and NO were $0.05 + x = 0.1022$ M each.

Hence, moles of each gas NO_2 and NO taken initially $= 5 \times 0.1022 = 0.511$

23. Pressure of the system will not influenced the rate of a chemical reaction involving liquids.

24. The two rates are related as $-\frac{1}{5}\frac{d[Br^-]}{dt} = \frac{1}{3}\frac{d[Br_2]}{dt}$

$$-\frac{d[Br^-]}{dt} = \frac{5}{3}\frac{d[Br_2]}{dt} = \frac{5}{3} \times 0.025 = 0.042 \text{ Ms}^{-1}$$

25. $LiAlH_4$ will reduce aldehyde further to alcohol.

26. Condensation of NH_2NH_2 at both nitrogens with C_6H_5CHO gives the desired product.

27. Acetaldehyde contains α-H, so it undergoes aldol condensation rather than Cannizzaro reaction.

28. Sucrose does not have free hemiacetal group, so does not undergo mutarotation.

29. Anomers differs in configuration only at C-1 carbons, they are simultaneously diastereomers.

30. 1 and 3 are C-2 epimers. C-2 epimers give same osazone on treatment with phenyl hydrazine.

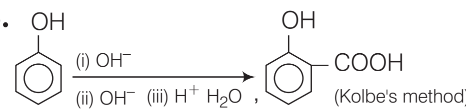
31. The tertiary structure of proteins refers to three dimensional folding of polymer chain.

32. Rubber has isotactic stereochemistry, not syndiotactic.

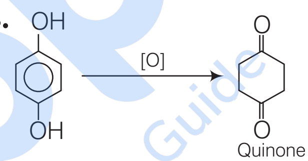
33. $$n\underset{\text{Acrylonitrile}}{CH_2 = \underset{|}{CH}(CN)} \xrightarrow{\text{Polymerisation}} \underset{\text{Polyacrylonitrile}}{\left(CH_2 - \underset{|}{CH}(CN) - \right)_n}$$

34. A nitro from *para* positions activate more for S_NAr reaction due to electron withdrawing resonance effect than two nitro from *meta* positions by inductive effect.

35. Ethyl in *o*/*p* directing group. Direct chlorination of ethyl benzene with $Cl_2/FeCl_3$ would give mixture of 2 and 4-chloroethyl benzene.

36. 

37. An aryl halide with $-NO_2$ substituents are highly activated for aromatic nucleophilic substitution reaction.

38. 

39. In physical adsorption, the adsorbent and adsorbate are held together by the van der Waals' forces.

40. Potassium stearate is an example of micelle.

41. Mass of Au deposited $=$ Number of Faraday passed $\times$ Eq. mass

$$= \frac{0.30 \times 15 \times 60}{96485} \times \frac{197}{3} = 0.184 \text{ g}$$

42. $E^0_{cell} = E^0_{cathode} - E^0_{anode}$

$$= 0.40 - (-0.44) = 0.84 \text{ V}$$

43. $$\underset{(1-\alpha)}{nA} \rightleftharpoons \underset{\frac{\alpha}{n}}{An}$$

$$i = 1 - \alpha + \frac{\alpha}{n}$$

44. Osmotic pressure $= iCRT$, For both the solutions, all other conditions are similar except 'i' which is 1.0 for sucrose while 2.0 for NaCl.

Therefore, osmotic pressure of NaCl

$$= 2 \times 0.25$$

$$= 0.50 \text{atm}$$

45. Mass of X decay $= 0.75\,g$

Moles of X decayed $= \dfrac{0.75}{253}$

$= $ moles of He formed

V(mL) of He at STP $= \dfrac{0.75 \times 22400}{253}$

$= 66.40$ mL

46. $K \propto \dfrac{1}{t}$

$$\log \frac{k_2}{k_1} = \log \frac{t_1}{t_2} = \frac{E_a}{R}\left(\frac{T_2 - T_1}{T_1 \cdot T_2}\right)$$

$$\log \frac{48}{4} = \frac{E_a}{8.314}\left(\frac{10}{300 \times 290}\right)$$

$$E_a = 78.0 \text{ kJ mol}^{-1}$$

47. mmol of H^+ (initial)$= 200 \times 0.031 \times 2 = 12.4$

mmol of OH^- (initial)$= 84 \times 0.15 = 12.6$

mmol of OH^- (left) after neutralisation $= 0.2$

$[OH^-]_{final} = \dfrac{0.2}{284} = 7 \times 10^{-4}$M

pOH $= 3.15$ and pH $= 10.85$

48. In $(SiH_3)_3N$, lone pair of the nitrogen is transferred to the empty d-orbitals of silicon ($p\pi - d\pi$ overlap) thereby causing planarity of unit.

49. Solid N_2O_5 is ionic. X-ray diffraction shows that solid N_2O_5 exists as $NO_2^+NO_3^-$ (nitronium nitrate).

50. White phosphorus is highly reactive, bursting into flames when exposed to air, and is thus stored under water.

51. The activity of alkaline earth metals as reducing agent increases from Be to Ba.

52. An improved combination of baking powder contains about 40% starch, 30% $NaHCO_3$, 20% $NaAl(SO_4)_2$ and 10% $Ca(H_2PO_4)_2$. Here starch acts as filler.

53. If acid is weak, its conjugate base is strong. Greater the oxidation number of Cl, stronger the acid and thus, weaker the conjugate base.

$$\underset{+1}{HClO}\quad \underset{+3}{HClO_2}\quad \underset{+5}{HClO_3}\quad \underset{+7}{HClO_4}$$
$$\longrightarrow$$

Oxidation number of Cl and acid strength increases

$$ClO^-\quad ClO_2^-\quad ClO_3^-\quad ClO_4^-$$
$$\longleftarrow$$

Conjugate base strength increases.

Hence, ClO^- is the strongest base.

54. I. $\underset{-3}{NH_4^+}\,\underset{+5}{NO_3^-} \longrightarrow \underset{+1}{N_2O} + 2H_2O$

It is a redox reaction but not a disproportionation reaction. Since, different N-species are involved.

II. $\underset{-3}{NH_4^+}\,\underset{+3}{NO_2^-} \longrightarrow \underset{0}{N_2}$

same as (I)

III. $\underset{+5\,-1}{PCl_5} \longrightarrow \underset{+3\,-1}{PCl_3} + \underset{0}{Cl_2}$

Thus, it is also not a disproportionation reaction.

55. $Fe^{2+} + e^- \longrightarrow Fe^+$

$NO \longrightarrow NO^+ + e^-$

Based on magnetic properties of the complex, it is found that iron has three unpaired electrons thus it exists as Fe^+ formed by reduction of Fe^{2+} by NO which is oxidised to NO^+.

56. Li – Mg $\rightarrow$ diagonal relationship.

B — Al, Na — K and Ca — Mg $\rightarrow$ belongs to same group.

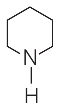
57. Along a group Z (effective nuclear charge) is almost constant. Thus, IE depends mainly on r_n (atomic radius).

58. Valence electron in $A = 3$

Valence electron in $B = 6$

Thus, A is electropositive and B is electronegative. A can lose three electrons and B can gain two electrons to attain stable configuration.

Thus, A exists as A^{3+} and B as B^{2-} thus, compound is A_2B_3.

59. In  lone pair of nitrogen is not involved in resonance as well as it has less s-character than the nitrogen of pyridine, hence the former is better electron donor.

60. Hyperconjugation involve delocalisation of σ-electrons with adjacent pi-system.

Mathematics

1. Given, $a = \cos\dfrac{2\pi}{7} + i.\sin\dfrac{2\pi}{7}$

$\therefore \quad a^7 = \cos 2\pi + i\sin 2\pi$

$[\because e^{i\theta} = \cos\theta + i\sin\theta]$

$= 1$

Also, $\alpha = a + a^2 + a^4,\ \beta = a^3 + a^5 + a^6$

then the sum of roots,

$$S = \alpha + \beta = a + a^2 + a^3 + a^4 + a^5 + a^6$$

$$\Rightarrow \quad S = \frac{a(1-a^6)}{1-a} = \frac{a-a^7}{1-a}$$

$$= \frac{a-1}{1-a} = -1 \qquad [\because a^7 = 1]$$

Product of the roots,

$P = \alpha\beta = (a + a^2 + a^4)(a^3 + a^5 + a^6)$

$= a^4 + a^5 + 1 + a^6 + 1 + a^2 + 1 + a + a^3\ [\because a^7 = 1]$

$= 3 + (a + a^2 + a^3 + a^4 + a^5 + a^6) = 3 - 1 = 2$

Hence, the required quadratic equation is

$$x^2 + x + 2 = 0$$

2. Let $x = (1)^{1/n}$

$\Rightarrow \quad x^n - 1 = 0$

or $\quad x^n - 1 = (x-1)(x-\omega)(x-\omega^2)\ldots(x-\omega^{n-1})$

$$\Rightarrow \quad \frac{x^n - 1}{x-1} = (x-\omega)(x-\omega^2)(x-\omega^3) \ldots (x-\omega^{n-1})$$

Putting $x = 9$ on both sides, we get

$$(9-\omega)(9-\omega^2)(9-\omega^3)\ldots(9-\omega^{n-1}) = \frac{9^n - 1}{8}$$

3. The given expression is

$$\left(x^2 + \frac{1}{x^2} + 2\right)^n = \left[\left(x + \frac{1}{x}\right)^2\right]^n$$

$$= \left(x + \frac{1}{x}\right)^{2n}$$

The binomial contains $(2n+1)$ terms in its expansion.

$\therefore$ The middle term is T_{n+1}.

$$T_{n+1} = {}^{2n}C_n\,(x)^{2n-n}\left(\frac{1}{x}\right)^n.$$

$$= {}^{2n}C_n$$

$$= \frac{(2n)!}{(2n-n)!\,n!} = \frac{(2n)!}{(n!)^2}$$

4. The equation of the tangent to parabola $y^2 = 4x$ is $y = mx + \dfrac{1}{m}$

This is also the tangent to parabola $x^2 = -8y$

$$\therefore \qquad x^2 = -8\left(mx + \frac{1}{m}\right)$$

$\Rightarrow mx^2 + 8m^2x + 8 = 0$

It has equal roots, if

$$64m^4 = 32m \qquad [\because b^2 = 4ac]$$

$$\Rightarrow \qquad m^3 = \frac{1}{2} \Rightarrow m = \frac{1}{\sqrt[3]{2}}$$

Hence, equation of the tangent is $y = \dfrac{1}{\sqrt[3]{2}}x + \sqrt[3]{2}$.

5. There may be two cases.

***Case* I.** Two children get none and one get three and others get one each.

Then, total number. of ways $= \dfrac{10!}{2! \times 3! \times 7!} \times 10!$

***Case* II.** Two get none and two get 2 each and the others get one each.

Then, total number of ways $= \dfrac{10!}{(2!)^4 \times 6!} \times 10!$

Hence, total number of ways

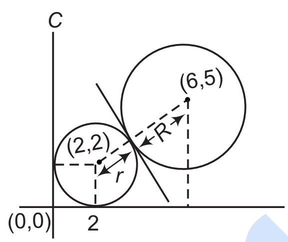
$$= \frac{(10!)^2}{2! \times 3! \times 7!} + \frac{(10!)^2}{(2!)^4 \times 6!} = \frac{(10!)^2 \times 25}{(2!)^2 \times 6! \times 84}$$

6. Variance of first n natural numbers is given by

$$\text{Variance} = \frac{\Sigma n^2}{n} - \left(\frac{\Sigma n}{n}\right)^2$$
$$= \frac{n(n+1)(2n-1)}{6n} - \left(\frac{n(n+1)}{2n}\right)^2$$
$$= (n+1)\left[\frac{2n-1}{6} - \frac{(n+1)}{4}\right]$$
$$= \frac{(n+1)}{12}[4n-2-3n-3]$$
$$= \frac{(n+1)}{12} \times (n-5)$$

7. Radius of smaller circle is $r = 2$ cm.

Since, it is touching both the axes, so its centre is (2,2).



The distance between their centres

$$= r + R = \sqrt{(5-2)^2 + (6-2)^2}$$
$$= \sqrt{3^2 + 4^2} = \sqrt{25} = 5 \text{ units}$$

8. Set $S = \{1, 2, 3, \ldots, 20\}$ is to be partitioned into four sets of equal size *i.e.*, having 5-5 numbers.

The number of ways in which it can be done

$$= {}^{20}C_5 \times {}^{15}C_5 \times {}^{10}C_5 \times {}^{5}C_5$$
$$= \frac{20!}{15! \times 5!} \times \frac{15!}{10! \times 5!} \times \frac{10!}{5! \times 5!} \times 1 = \frac{(20!)}{(5!)^4}$$

9. Total number of cases $= {}^{50}C_1 = 50$

Let A be the event of selecting ticket with sum of digits '8'.

Favourable cases to A are $\{08, 17, 26, 35, 44\}$.

Let B be the event of selecting ticket with product of its digits '7'

Favourable cases to B is only $\{17\}$.

$$\text{Now, } P(B/A) = \frac{P(A \cap B)}{P(A)}$$
$$= \frac{1/50}{5/50} = \frac{1}{5}$$

10. General term in the expansion of $\left(\sqrt{x} - \frac{k}{x^2}\right)^{10}$ is

$$T_{r+1} = {}^{10}C_r (\sqrt{x})^{10-r} \cdot \left(\frac{-k}{x^2}\right)^r$$
$$= {}^{10}C_r x^{\frac{10-r}{2}} \cdot (-k)^r \cdot x^{-2r}$$
$$= {}^{10}C_r (-k)^r x^{\left(\frac{10-5r}{2}\right)}$$

$\because$ The term is free from x

$\therefore$ Put $\frac{10-5r}{2} = 0$

$\Rightarrow$ $r = 2$

Now, ${}^{10}C_2 (-k)^2 = 405$

$\Rightarrow$ $\frac{10 \times 9}{1 \times 2} \cdot k^2 = 405$

$\Rightarrow$ $k^2 = \frac{405}{45} = 9$

$\Rightarrow$ $k = \pm 3$

11. The given equation is $8x^2 - 26x + 15 = 0$.

$\therefore$ The sum of roots,

$\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} = \frac{26}{8} = \frac{13}{4}$ and product of roots,

$\tan\frac{\alpha}{2} \cdot \tan\frac{\beta}{2} = \frac{15}{8}$

$$\therefore \tan\left(\frac{\alpha+\beta}{2}\right) = \frac{\tan\frac{\alpha}{2} + \tan\frac{\beta}{2}}{1 - \tan\frac{\alpha}{2} \cdot \tan\frac{\beta}{2}}$$
$$= \frac{\frac{13}{4}}{1 - \frac{15}{8}} = -\frac{26}{7}$$

$$\text{Now, } \cos(\alpha + \beta) = \frac{1 - \tan^2\left(\frac{\alpha+\beta}{2}\right)}{1 + \tan^2\left(\frac{\alpha+\beta}{2}\right)}$$

$$\left[\because \cos 2\theta = \frac{1 - \tan^2\theta}{1 + \tan^2\theta}\right]$$

$$= \frac{1 - \left(-\frac{26}{7}\right)^2}{1 + \left(-\frac{26}{7}\right)^2} = \frac{49 - 676}{49 + 676} = -\frac{627}{725}$$

12. Given relations are

$$a\cos^3\alpha + 3a\cos\alpha\sin^2\alpha = m \quad \ldots(i)$$

$$a\sin^3\alpha + 3a\cos^2\alpha\sin\alpha = n \quad \ldots(ii)$$

On adding Eq. (i) and (ii) we get

$$(m+n) = a\cos^3\alpha + 3a\cos\alpha\sin^2\alpha + 3a\cos^2\alpha\sin\alpha + a\sin^3\alpha$$

$$= a(\cos\alpha + \sin\alpha)^3$$

Similarly on subtracting Eq. (ii) from Eq. (i), we get

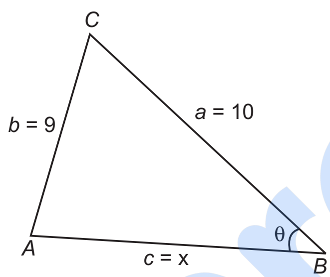
$$m - n = a(\cos\alpha - \sin\alpha)^3$$

Now, $(m+n)^{2/3} + (m-n)^{2/3}$

$$= a^{2/3}\{(\cos\alpha + \sin\alpha)^2 + (\cos\alpha - \sin\alpha)^2\}$$

$$= a^{2/3}\{2(\cos^2\alpha + \sin^2\alpha)\} = 2a^{2/3}$$

13. Let in the figure, $\angle A > \angle B > \angle C$



$\because$ Angles are in A.P.

$\therefore \quad \angle C = \theta - d, \angle B = \theta$

and $\quad \angle A = \theta + d$

As $\quad \angle A + \angle B + \angle C = 180°$

$\Rightarrow \quad \theta + d + \theta + \theta - d = 180°$

$\Rightarrow \quad 3\theta = 180°$

$\Rightarrow \quad \theta = 60°$

Now in ΔABC,

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos 60° = \frac{(10)^2 + x^2 - 9^2}{2 \times 10 \times x}$$

$$\Rightarrow \quad \frac{1}{2} = \frac{100 + x^2 - 9^2}{20x}$$

$$\Rightarrow \quad x^2 - 10x + 19 = 0$$

$$\Rightarrow \quad x = \frac{10 \pm \sqrt{100 - 76}}{2}$$

$$= \frac{10 \pm \sqrt{24}}{2} = 5 \pm \sqrt{6}$$

14. Let the equation of the line parallel to $x - 2y = 1$ is $x - 2y + \lambda = 0$

Since, it passes through (3, 5)

$\therefore \quad 3 - 10 + \lambda = 0 \Rightarrow \lambda = 7$

$\therefore$ The line is $x - 2y + 7 = 0$.

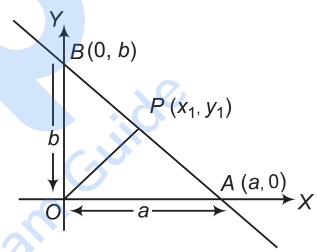
The point of intersection of $x - 2y + 7 = 0$ and $2x + 3y - 14 = 0$ is (1, 4).

$\therefore$ The distance between (3, 5) and (1, 4)

$$= \sqrt{(3-1)^2 + (5-4)^2} = \sqrt{4+1} = \sqrt{5}$$

15. Equation of line is $\dfrac{x}{a} + \dfrac{y}{b} = 1 \quad \ldots(i)$

Let the foot of the perpendicular drawn from the origin to the line be $P(x_1, y_1)$.



Since, $OP \perp AB$

$\therefore$ slope of $OP \times$ slope of $AB = -1$

$$\Rightarrow \quad \frac{y_1}{x_1} \times \frac{b}{-a} = -1 \Rightarrow by_1 = ax_1 \quad \ldots(ii)$$

Since, P lies on the line AB, so

$$\frac{x_1}{a} + \frac{y_1}{b} = 1$$

$$\Rightarrow \quad bx_1 + ay_1 = ab \quad \ldots(iii)$$

From (ii) and (iii), we get

$$x_1 = \frac{ab^2}{a^2 + b^2} \quad \text{and} \quad y_1 = \frac{a^2 b}{a^2 + b^2}$$

Now, $\quad x_1^2 + y_1^2 = \left(\dfrac{ab^2}{a^2+b^2}\right)^2 + \left(\dfrac{a^2 b}{a^2+b^2}\right)^2$

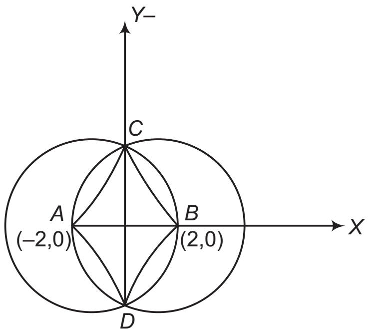
$$= \frac{a^2 b^4}{(a^2+b^2)^2} + \frac{a^4 b^2}{(a^2+b^2)^2} = \frac{a^2 b^2}{(a^2+b^2)^2}(a^2+b^2)$$

$$= \frac{a^2 b^2}{a^2 + b^2} = \frac{1}{\dfrac{1}{a^2} + \dfrac{1}{b^2}}$$

$$\Rightarrow \quad x_1^2 + y_1^2 = \frac{1}{c^2} \qquad \left(\because \frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}\right)$$

Thus, the locus of $P(x_1, y_1)$ is $x^2 + y^2 = c^2$, which is a circle.

16. Circles with centres $A(-2, 0)$ and $B(+2, 0)$, each of radius 4 has Y-axis as their common chord.



$\Rightarrow \Delta\ ABC$ is an equilateral triangle.

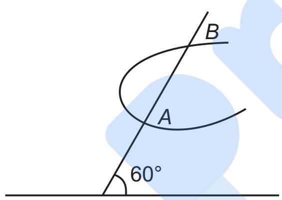
Hence, the area of rhombus $ADBC$ is

$$= 2 \cdot \frac{\sqrt{3}}{4} \times 4^2 = 8\sqrt{3}$$

17. Given $P = (\sqrt{3}, 0)$

$\therefore$ Equation of line AB is

$$\frac{x - \sqrt{3}}{\cos 60°} = \frac{y - 0}{\sin 60°} = r \quad \text{(say)}$$



$$\Rightarrow \quad x = \sqrt{3} + \frac{r}{2},\ y = \frac{r\sqrt{3}}{2}$$

$\therefore$ point $\left(\sqrt{3} + \frac{r}{2}, \frac{r\sqrt{3}}{2}\right)$ lies on $y^2 = x + 2$

$$\therefore \quad \frac{3r^2}{4} = \sqrt{3} + \frac{r}{2} + 2$$

$$\Rightarrow \quad \frac{3r^2}{4} - \frac{r}{2} - (2 + \sqrt{3}) = 0$$

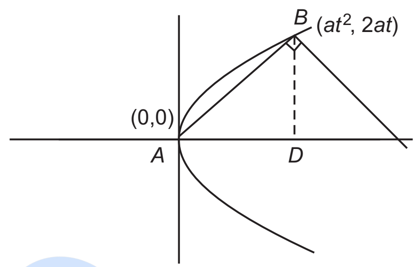
Let the roots be r_1 and r_2, then the product

$$r_1 \times r_2 = PA \, . \, PB = \left| \frac{-(2 + \sqrt{3})}{\frac{3}{4}} \right|$$

$$= \frac{4(2 + \sqrt{3})}{3}$$

18. Given equation of parabola is $y^2 = 4ax$.

Let the co-ordinates of B are $(at^2, 2at)$, then the slope of $AB = \frac{2}{t}$



Since, $BC \perp AB$

$\therefore$ Slope of $BC = \frac{-t}{2}$

The equation of BC is

$$y - 2 = -\frac{t}{2}(x - at^2)$$

$\because$ This line meets the X-axis at point C.

So, put $y = 0$, we get

$$\Rightarrow \quad x = 4a + at^2$$

So, the distance $CD = 4a + at^2 - at^2$

$$= 4a$$

19. Given equation of ellipse is $\frac{x^2}{4} + \frac{y^2}{\frac{7}{4}} = 1$

Here, $\quad a^2 = 4$

and $\quad b^2 = \frac{7}{4}$

$\because \quad b^2 = a^2(1 - e^2)$

$\therefore \quad \frac{7}{4} = 4(1 - e^2)$

$\Rightarrow \quad e^2 = 1 - \frac{7}{16} = \frac{9}{16} \Rightarrow e = \frac{3}{4},$

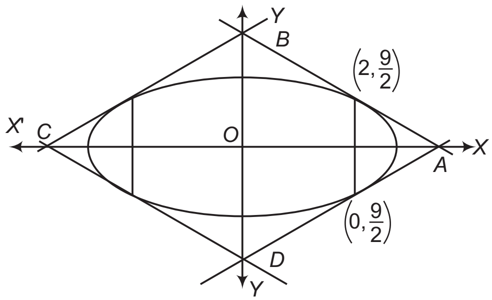
Thus, the foci are $\left(\pm \frac{3}{2}, 0\right)$.

The radius of the required circle

$$= \sqrt{\left(\frac{3}{2} - \frac{1}{2}\right)^2 + (0 - 2)^2}$$

$$= \sqrt{5}$$

20. The quadrilateral formed by the tangents at the end points of the latusrectum is a rhombus. It is symmetrical about the axes.



So, total area is four times the area of the right triangle formed by the tangents and the axes in the first quadrant.

Now, $\quad ae = \sqrt{a^2 - b^2}$

$\Rightarrow \quad ae = 2$

$\therefore$ The co-ordinates of one end point of latusrectum are $\left(2, \frac{5}{3}\right)$.

The equation of tangent at that point is

$\frac{x}{\frac{9}{2}} + \frac{y}{3} = 1$, this meets the co-ordinate axes at $A\left(0, \frac{9}{2}\right)$ and B (3, 0).

Area of $\Delta AOB = \frac{1}{2} \times \frac{9}{2} \times 3 = \frac{27}{4}$

Hence, the area of rhombus $ABCD$

$$= 4 \times \frac{27}{4} = 27 \text{ sq units}$$

21. Equation of chord of hyperbola $x^2 - y^2 = a^2$ with mid-point as (h, k) is given by

$$xh - yk = h^2 - k^2$$

$$\Rightarrow \quad y = \frac{h}{k}x - \frac{(h^2 - k^2)}{k}$$

This will touch the parabola $y^2 = 4ax$, if

$$-\left(\frac{h^2 - k^2}{k}\right) = \frac{a}{\frac{h}{k}}$$

$$\Rightarrow \quad ak^2 = h^3 + k^2 h$$

$\therefore$ Locus of the mid-point is $x^3 = y^2 (x - a)$

22. $\lim_{x \to 0} \frac{e^{\sin x} - 1}{x}$

$$= \lim_{x \to 0} \left[\frac{(e^{\sin x} - 1)}{\sin x} \times \frac{\sin x}{x}\right]$$

$$= \lim_{\sin x \to 0} \frac{e^{\sin x} - 1}{\sin x} \times \lim_{x \to 0} \frac{\sin x}{x}$$

$$= 1 \times 1 = 1$$

23. By truth table

p	q	$(p \wedge q)$	$\sim(p \wedge q)$	$q \Leftrightarrow p$	$\sim(q \Leftrightarrow p)$	$\sim(p \wedge q) \vee \sim(q \Leftrightarrow p)$
T	T	T	F	T	F	F
T	F	F	T	F	T	T
F	T	F	T	F	T	T
F	F	F	T	T	F	T

It is clear that the given statement is neither tautology nor contradiction.

24. Given determinant is

$$\Delta = \begin{vmatrix} (a^x + a^{-x})^2 & (a^x - a^{-x})^2 & 1 \\ (b^x + b^{-x})^2 & (b^x - b^{-x})^2 & 1 \\ (c^x + c^{-x})^2 & (c^x - c^{-x})^2 & 1 \end{vmatrix}$$

Applying $C_1 \to C_1 - C_2$, we get

$$\Delta = \begin{vmatrix} 4 & (a^x - a^{-x})^2 & 1 \\ 4 & (b^x - b^{-x})^2 & 1 \\ 4 & (c^x - c^{-x})^2 & 1 \end{vmatrix} = 4 \begin{vmatrix} 1 & (a^x - a^{-x})^2 & 1 \\ 1 & (b^x - b^{-x})^2 & 1 \\ 1 & (c^x - c^{-x})^2 & 1 \end{vmatrix}$$

$= 0 \qquad$ ($\because$ two columns are identical)

25. Putting $r = 1, 2, 3, \ldots, n$ and using the formula

$$\Sigma 1 = n$$

and $\quad \Sigma r = \frac{n(n+1)}{2}$

$$\Sigma(2r - 1) = 1 + 3 + 5 + \ldots = n^2$$

$$\therefore \sum_{r=1}^{n} \Delta_r = \begin{vmatrix} n & n & n \\ n(n+1) & n^2 + n + 1 & n^2 + n \\ n^2 & n^2 & n^2 + n + 1 \end{vmatrix} = 56$$

Applying $C_1 \to C_1 - C_3, C_2 \to C_2 - C_3$, we get

$$\begin{vmatrix} 0 & 0 & n \\ 0 & 1 & n^2 + n \\ -n - 1 & -n - 1 & n^2 + n + 1 \end{vmatrix} = 56$$

$$\Rightarrow \quad n(n+1) = 56$$

$$\Rightarrow \quad n^2 + n - 56 = 0$$

$$\Rightarrow \quad (n+8)(n-7)=0$$

$$\Rightarrow \quad n=7, (n \neq -8)$$

26. $$\Delta = \begin{vmatrix} e^a & e^{2a} & e^{3a} \\ e^b & e^{2b} & e^{3b} \\ e^c & e^{2c} & e^{3c} \end{vmatrix} - \begin{vmatrix} e^a & e^{2a} & 1 \\ e^b & e^{2b} & 1 \\ e^c & e^{2c} & 1 \end{vmatrix}$$

$$= e^a \cdot e^b \cdot e^c \begin{vmatrix} 1 & e^a & e^{2a} \\ 1 & e^b & e^{2b} \\ 1 & e^c & e^{2c} \end{vmatrix} + \begin{vmatrix} e^a & 1 & e^{2a} \\ e^b & 1 & e^{2b} \\ e^c & 1 & e^{2c} \end{vmatrix}$$

$$= e^{(a+b+c)} \begin{vmatrix} 1 & e^a & e^{2a} \\ 1 & e^b & e^{2b} \\ 1 & e^c & e^{2c} \end{vmatrix} - \begin{vmatrix} 1 & e^a & e^{2a} \\ 1 & e^b & e^{2b} \\ 1 & e^c & e^{2c} \end{vmatrix}$$

$$= (e^{a+b+c}-1) \begin{vmatrix} 1 & e^a & e^{2a} \\ 1 & e^b & e^{2b} \\ 1 & e^c & e^{2c} \end{vmatrix}$$

$$= 0 \qquad (\because a+b+c=0)$$

27. $fog = f\{g(x)\} = |\sin x| = \sqrt{\sin^2 x}$

Also, $gof = g\{f(x)\} = \sin^2 \sqrt{x}$

Obviously, $\sqrt{\sin^2 x} = \sqrt{g(x)}$

and $\sin^2 \sqrt{x} = \sin^2\{f(x)\}$

i.e., $g(x) = \sin^2 x$

and $f(x) = \sqrt{x}$

28. Given, $f(x) = \dfrac{x}{\sqrt{1+x^2}}$

$$(fof)x = \{f(f(x))\}$$

$$= f\left(\frac{x}{\sqrt{1+x^2}}\right) = \frac{\frac{x}{\sqrt{1+x^2}}}{\sqrt{1+\frac{x^2}{1+x^2}}}$$

$$\Rightarrow \quad (fof)x = \frac{x}{\sqrt{1+2x^2}}$$

Now, $(fofof)x = f\{(fof)x\}x = f\left(\dfrac{x}{\sqrt{1+2x^2}}\right)$

$$= \frac{\frac{x}{\sqrt{1+2x^2}}}{\sqrt{1+\frac{x^2}{1+2x^2}}} = \frac{x}{\sqrt{1+3x^2}}$$

29. On putting $y=0$, in the given relation, we get

$f(x)+f(x) = k \cdot f(x) \cdot f(0)$

$$\Rightarrow \quad 2f(x) = k \cdot f(x) \qquad [\because f(0)=1]$$

$$\therefore \quad k=2$$

30. Given, $\tan^{-1}\dfrac{a}{x} + \tan^{-1}\dfrac{b}{x} = \dfrac{\pi}{2}$

$$\Rightarrow \quad \tan^{-1}\left[\frac{\left(\frac{a}{x}+\frac{b}{x}\right)}{1-\frac{ab}{x^2}}\right] = \frac{\pi}{2}$$

$$\Rightarrow \quad \frac{\frac{a+b}{x}}{\frac{x^2-ab}{x^2}} = \tan\frac{\pi}{2}$$

$$\Rightarrow \quad x^2 = ab$$

$$\Rightarrow \quad x = \sqrt{ab}$$

31. $\cos^{-1}[\cos\{2 \cdot \cot^{-1}(\sqrt{2}-1)\}]$

$$= \cos^{-1}[\cos\{2 \times 67.5\}]$$

$$= \cos^{-1}[\cos 135^\circ]$$

$$= 135^\circ$$

$$= \frac{3\pi}{4}$$

32. We have,

$$\sum_{m=1}^{n} \tan^{-1}\left(\frac{2m}{m^4+m^2+2}\right)$$

$$= \sum_{m=1}^{n} \tan^{-1}\left[\frac{2m}{1+(m^2+m+1)(m^2-m+1)}\right]$$

$$= \sum_{m=1}^{n} \tan^{-1}\left[\frac{(m^2+m+1)-(m^2-m+1)}{1+(m^2+m+1)(m^2-m+1)}\right]$$

$$= \sum_{m=1}^{n} \{\tan^{-1}[m^2+m+1] - \tan^{-1}[m^2-m+1]\}$$

$$= \tan^{-1}3 - \tan^{-1}1 + \tan^{-1}7 - \tan^{-1}3$$
$$+ (\tan^{-1}13 - \tan^{-1}7) + \ldots + \tan^{-1}(n^2+n+1)$$
$$- \tan^{-1}(n^2-n+1)$$

$$= \tan^{-1}\left(\frac{n^2+n+1-1}{1+(n^2+n+1)\cdot 1}\right) = \tan^{-1}\left(\frac{n^2+n}{2+n^2+n}\right)$$

33. Put $x = \frac{\pi}{2} - h$ as $x \to \frac{\pi}{2}, h \to 0$

$\therefore$ Given, limit $= \lim_{h \to 0} \frac{1 - \tan\left(\frac{\pi}{4} + \frac{h}{2}\right)(1 - \cos h)}{1 + \tan\left(\frac{\pi}{4} + \frac{h}{2}\right)(2h)^3}$

$$= \lim_{h \to 0} \tan\left(\frac{h}{2}\right) \lim_{h \to 0} \frac{2 \cdot \sin^2\left(\frac{h}{2}\right)}{8h^3}$$

$$= \lim_{h \to 0} \frac{1}{4}\left(\frac{\tan \frac{h}{2}}{2 \times \frac{h}{2}}\right) \times \lim_{h \to 0} \left(\frac{\sin \frac{h}{2}}{\frac{h}{2}}\right)^2 \times \frac{1}{4}$$

$$= \frac{1}{32}$$

34. We have,

$$f(x) = \begin{cases} \frac{\sin^3(\sqrt{x}) \cdot \log(1 + 3x)}{x \cdot (\tan^{-1}\sqrt{x})^2 \cdot (e^{5\sqrt{x}} - 1)}, & x \neq 0 \\ a, & x = 0 \end{cases}$$

For continuity in $[0, 1]$, $f(0) = f(0^+)$, otherwise it is discontinuous.

$$\therefore \lim_{x \to 0^+} \frac{\sin^3(\sqrt{x}) \cdot \log(1 + 3x)}{x \cdot (\tan^{-1}\sqrt{x})^2 \cdot (e^{5\sqrt{x}} - 1)}$$

$$= \lim_{x \to 0^+} \left[\frac{3}{5} \cdot \frac{\sin^3\sqrt{x}}{(\sqrt{x})^3} \cdot \frac{(\sqrt{x})^2}{(\tan^{-1}\sqrt{x})^2} \times \frac{\log(1 + 3x)}{3x} \cdot \frac{5\sqrt{x}}{e^{5\sqrt{x}} - 1}\right]$$

$$= \frac{3}{5} \lim_{x \to 0^+} \frac{\sin^3\sqrt{x}}{(\sqrt{x})^3} \cdot \frac{(\sqrt{x})^2}{\tan^{-1}\sqrt{x}} \times \frac{\log(1 + 3x)}{3x} \cdot \frac{5\sqrt{x}}{e^{5\sqrt{x}} - 1} = a$$

$$\Rightarrow \quad a = \frac{3}{5}$$

35. Given $y = |\sin x|^{|x|}$

In the neighbourhood of $-\frac{\pi}{6}$, $|x|$ and $|\sin x|$ both are negative

i.e., $y = (-\sin x)^{-x}$

Taking log on both sides, we get

$\log y = -x \log(-\sin x)$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = (-x)\frac{1}{-\sin x}(-\cos x) + \log(-\sin x)(-1)$$

$$= -x \cdot \cot x - \log(-\sin x)$$

$$= -[x \cot x + \log(-\sin x)]$$

$$\Rightarrow \quad \frac{dy}{dx} = -y[x \cot x + \log(-\sin x)]$$

$$\therefore \left(\frac{dy}{dx}\right)_{x = -\frac{\pi}{6}} = (2)^{-\frac{\pi}{6}} \frac{[6\log 2 - \sqrt{3}\pi]}{6}$$

36. Let $y = x^x$

$$\Rightarrow \frac{dy}{dx} = x^x(1 + \log x)$$

For increasing function, $\frac{dy}{dx} > 0$

$\Rightarrow \quad x^x (1 + \log x) > 0$

$\Rightarrow \quad 1 + \log x > 0$

$\Rightarrow \quad \log_e x > \log_e \frac{1}{e}$

$\Rightarrow \quad x > \frac{1}{e}$

$\therefore$ The function is increasing when $x > \frac{1}{e}$

37. Let $f(x) = 3\sin x - 4\sin^3 x = \sin 3x$

Since, $\sin x$ is increasing in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$\therefore \quad -\frac{\pi}{2} \leq 3x \leq \frac{\pi}{2} \quad \Rightarrow -\frac{\pi}{6} \leq x \leq \frac{\pi}{6}$$

Thus, the length of interval $= \left[\frac{\pi}{6} - \left(-\frac{\pi}{6}\right)\right] = \frac{\pi}{3}$

38. Given curve is $y = f(x) = x^2 + bx - b$

On differentiating w.r.t.x, we get

$$\frac{dy}{dx} = 2x + b$$

The equation of the tagent at $(1, 1)$ is

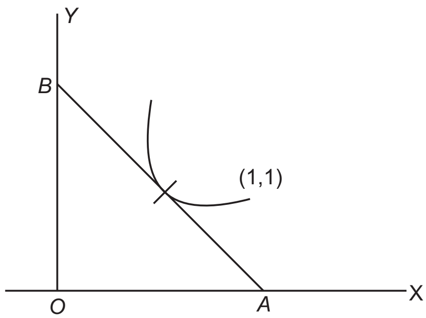
$$y - 1 = \left(\frac{dy}{dx}\right)_{(1, 1)} (x - 1)$$

$\Rightarrow \quad y - 1 = (b + 2)(x - 1)$

$\Rightarrow \quad (2 + b)x - y = 1 + b$

$$\Rightarrow \frac{x}{(1 + b)/(2 + b)} - \frac{y}{(1 + b)} = 1$$

So, $OA = \frac{1 + b}{2 + b}$ and $OB = -(1 + b)$



Now, area of ΔAOB

$$= \frac{1}{2}\frac{(1+b)}{(2+b)}[-(1+b)] = 2 \quad \text{(given)}$$

$$\Rightarrow \quad 4(2+b) + (1+b)^2 = 0$$

$$\Rightarrow \quad 8 + 4b + 1 + b^2 + 2b = 0$$

$$\Rightarrow \quad b^2 + 6b + 9 = 0$$

$$\Rightarrow \quad (b+3)^2 = 0 \Rightarrow b = -3$$

39. The given curve is $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$

$$\Rightarrow \quad n\left(\frac{x}{a}\right)^{n-1}\cdot\frac{1}{a} + n\left(\frac{y}{b}\right)^{n-1}\cdot\frac{1}{b}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{b}{a}\frac{\left(\frac{x}{a}\right)^{n-1}}{\left(\frac{y}{b}\right)^{n-1}}$$

At (a, b), $\frac{dy}{dx} = -\frac{b}{a}\frac{(1)^{n-1}}{(1)^{n-1}} = -\frac{b}{a}$

$\therefore$ The tangent at (a, b) is

$$\Rightarrow \quad y - b = -\frac{b}{a}(x - a)$$

$$\Rightarrow \quad ay - ab = -bx + ab$$

$$\Rightarrow \quad bx + ay = 2ab$$

$$\Rightarrow \quad \frac{x}{a} + \frac{y}{b} = 2$$

40. Given curve is $\frac{a^2}{x^2} + \frac{b^2}{y^2} = 1$

Let radius vector is 'r'

$$\therefore \quad r^2 = x^2 + y^2$$

$$\Rightarrow \quad r^2 = \frac{a^2y^2}{y^2 - b^2} + y^2 \qquad \left(\because \frac{a^2}{x^2} + \frac{b^2}{y^2} = 1\right)$$

For minumum value of r,

$$\frac{d(r^2)}{dy} = 0$$

$$\Rightarrow \quad \frac{-2y\,b^2a^2}{(y^2 - b^2)^2} + 2y = 0$$

$$\Rightarrow \quad y^2 = b(a+b)$$

$$\therefore \quad x^2 = a(a+b)$$

$$\Rightarrow \quad r^2 = (a+b)^2$$

$$\Rightarrow \quad r = a + b$$

41. Consider the function $f(x) = \frac{x^2}{x^3 + 200}$

$$f'(x) = x\frac{(400 - x^3)}{(x^3 + 200)^2} = 0$$

$$\Rightarrow \quad x^3 = 400$$

$$\Rightarrow \quad x = (400)^{1/3}$$

When $x < 0$, $f'(x) > 0$

$x > 0$, $f'(x) < 0$

$\therefore$ $f(x)$ has maximum at $x = (400)^{1/3}$

Since, $7 < (400)^{1/3} < 8$, either a_7 or a_8 is the greatest term of the sequence.

$$\because \quad a_7 = \frac{49}{543},\ a_8 = \frac{8}{89}$$

and $\quad \frac{49}{543} > \frac{8}{89}$

$$a_7 = \frac{49}{543} \text{ is the greatest.}$$

42. $f(x) = \begin{cases} |x^3 + x^2 + 3x + \sin x|\left(3 + \sin\frac{1}{x}\right) & x \neq 0 \\ 0, & x = 0 \end{cases}$

Let $\quad g(x) = x^3 + x^2 + 3x + \sin x$

$$g'(x) = 3x^2 + 2x + 3 + \cos x$$

$$= 3\left(x^2 + \frac{2x}{3} + 1\right) + \cos x$$

$$\Rightarrow \quad g'(x) = 3\left\{\left(x + \frac{1}{3}\right)^2 + \frac{8}{9}\right\} + \cos x > 0$$

and $\; 2 < 3 + \sin\left(\frac{1}{x}\right) < 4$

Hence, minimum value of $f(x)$ is 0 at $x = 0$

Hence, the number of points $= 1$

43. Let, $I = \int \frac{x^2 - 1}{(x^4 + 3x^2 + 1)\tan^{-1}\left(x + \frac{1}{x}\right)} dx$

On dividing numerator and denominator by x^2, we get

$$I = \int \frac{1 - \frac{1}{x^2}}{\left(x^2 + \frac{1}{x^2} + 3\right)\tan^{-1}\left(x + \frac{1}{x}\right)} dx$$

Put $\quad x + \frac{1}{x} = t$

$\Rightarrow \left(1 - \frac{1}{x^2}\right)dx = dt$ and

$$x^2 + \frac{1}{x^2} = t^2 - 2$$

$\therefore$
$$I = \int \frac{dt}{(t^2 - 2 + 3)\tan^{-1} t}$$
$$= \int \frac{dt}{(1 + t^2)\tan^{-1} t}$$
$$= \log(\tan^{-1} t) + C$$
$$= \log\left[\tan^{-1}\left(x + \frac{1}{x}\right)\right] + C$$

44. Let, $I = \int \frac{\cos x}{\sin^2 x(\sin x + \cos x)} dx$

On dividing numerator and denominator by $\cos^3 x$, we get

$$I = \int \frac{\sec^2 x}{\tan^2 x\,(1 + \tan x)} dx$$

Put $\quad \tan x = t \quad \Rightarrow \quad \sec^2 x dx = dt$

$\therefore$
$$I = \int \frac{dt}{t^2(1+t)}$$
$$= \int \left[-\frac{1}{t} + \frac{1}{t^2} + \frac{1}{1+t}\right] dt$$

(using partial fraction)

$\Rightarrow$
$$I = -\int \frac{1}{t} dt + \int t^{-2} dt + \int \frac{1}{1+t} dt$$
$$= -\log|t| - \frac{1}{t} + \log|1 + t| + C$$
$$= -\frac{1}{\tan x} + \log\left|\frac{1 + \tan x}{\tan x}\right| + C$$
$$= -\cot x + \log\left|\frac{1 + \tan x}{\tan x}\right| + C$$

45. Let, $I = \int \frac{dx}{\sin x - \cos x + \sqrt{2}}$

$$\Rightarrow \quad I = \int \frac{dx}{\sqrt{2}\left(\frac{1}{\sqrt{2}}\sin x - \frac{1}{\sqrt{2}}\cos x\right) + \sqrt{2}}$$
$$= \frac{1}{\sqrt{2}} \int \frac{dx}{1 - \cos\left(x + \frac{\pi}{4}\right)} = \frac{1}{\sqrt{2}} \int \frac{dx}{2\sin^2\left(\frac{x}{2} + \frac{\pi}{8}\right)}$$
$$= \frac{1}{2\sqrt{2}} \int \frac{dx}{\sin^2\left(\frac{x}{2} + \frac{\pi}{8}\right)} = \frac{1}{2\sqrt{2}} \int \operatorname{cosec}^2\left(\frac{x}{2} + \frac{\pi}{8}\right) dx$$
$$= \frac{1}{2\sqrt{2}}\left[\frac{-\cot\left(\frac{x}{2} + \frac{\pi}{8}\right)}{\frac{1}{2}}\right] + C$$
$$= -\frac{1}{\sqrt{2}}\cot\left(\frac{x}{2} + \frac{\pi}{8}\right) + C$$

46. $\int_0^{\frac{3\pi}{2}} \sin\left(\frac{2x}{\pi}\right) dx = \int_0^{\frac{\pi}{2}} \sin\left(\frac{2x}{\pi}\right) dx + \int_{\frac{\pi}{2}}^{\pi} \sin\left(\frac{2x}{\pi}\right) dx + \int_{\pi}^{\frac{3\pi}{2}} \sin\left(\frac{2x}{\pi}\right) dx$

$$= 0 + \sin 1 \int_{\frac{\pi}{2}}^{\pi} dx + \sin 2 \int_{\pi}^{\frac{3\pi}{2}} dx$$
$$= \frac{\pi}{2}(\sin 1 + \sin 2)$$

47. Let, $I = \int_{\frac{1}{n}}^{(an-1)/n} \frac{\sqrt{x}}{\sqrt{a - x} + \sqrt{x}} dx$

$$= \int_{\frac{1}{n}}^{a - \frac{1}{n}} \frac{\sqrt{x}}{\sqrt{a - x} + \sqrt{x}} dx \quad \ldots(\text{i})$$

$$= \int_{\frac{1}{n}}^{a - \frac{1}{n}} \frac{\left(\sqrt{\frac{1}{n} + a - \frac{1}{n} - x}\right)}{\sqrt{a - \left(\frac{1}{n} + a - \frac{1}{n} - x\right)} + \sqrt{\frac{1}{n} + a - \frac{1}{n} - x}} dx$$

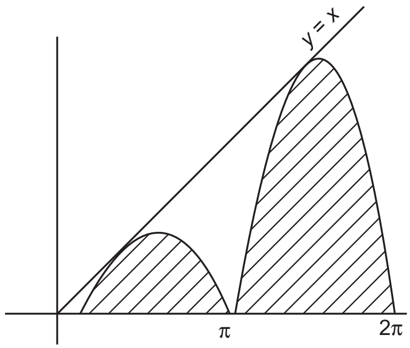
$$\Rightarrow \quad I = \int_{\frac{1}{n}}^{a - \frac{1}{n}} \frac{\sqrt{a - x}}{\sqrt{x} + \sqrt{a - x}} dx \quad \ldots(\text{ii})$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_{\frac{1}{n}}^{a - \frac{1}{n}} 1\,dx = [x]_{\frac{1}{n}}^{a - \frac{1}{n}}$$
$$= a - \frac{1}{n} - \frac{1}{n} = \frac{na - 2}{n} \Rightarrow I = \frac{na - 2}{2n}$$

48. $y = x|\sin x|$

$$\Rightarrow \quad y = \begin{cases} x\sin x, & 0 \le x \le \pi \\ -x\sin x, & \pi < x \le 2\pi \end{cases}$$



$\therefore$ Required Area $= \int_0^{\pi} x\sin x + \int_{\pi}^{2\pi}(-x\sin x)\,dx$

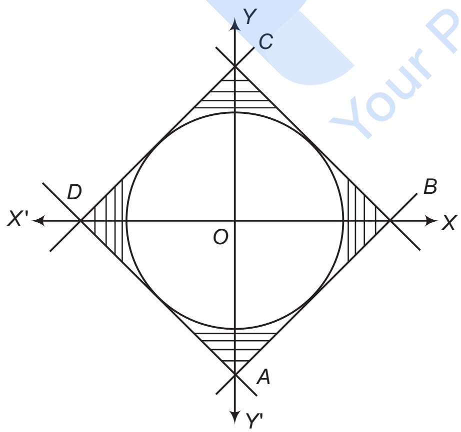
$$= [x(-\cos x)]_0^{\pi} - \int_0^{\pi}(-\cos x)\,dx$$

$$- [\{x(-\cos x)\}]_{\pi}^{2\pi} + \int_{\pi}^{2\pi}(-\cos x)\,dx$$

$$= [\pi + \sin x]_0^{\pi} + [(2\pi + \pi) - \sin x]_{\pi}^{2\pi}$$

$= 4\pi$ sq units

49. The curves $y + x \le 1$ and $|y - x| \le 1$ form a square $ABCD$ and $2x^2 + 2y^2 = 1$ is a circle of radius $\frac{1}{\sqrt{2}}$ and centre $(0, 0)$.



$\therefore$ Required Area

$=$ Area of square $-$ Area of circle

$= \left(2 - \frac{\pi}{2}\right)$ sq units

50. $y = u^m \Rightarrow \frac{dy}{dx} = m \cdot u^{m-1} \cdot \frac{du}{dx}$

Hence, $2x^4 \cdot mu^{m-1}u^m \cdot \frac{du}{dx} + u^{4m} = 4x^6$

$$\Rightarrow \quad \frac{du}{dx} = \frac{4x^6 - u^{4m}}{2mx^4 \cdot u^{2m-1}}$$

$$\therefore \quad 4m = 6 \Rightarrow m = \frac{3}{2}$$

51. Equation of the normal at (x_1, y_1) is

$$y - y_1 = \frac{dx}{dy}(x - x_1)$$

Put $y = 0$, then

$$x = x_1 + y_1\frac{dy}{dx}$$

$$y_1^2 = 2x_1 x$$

$$= 2x_1\left(x_1 + y_1\frac{dy}{dx}\right)$$

$$\Rightarrow \quad y_1^2 = 2x_1^2 + 2x_1y_1\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y_1^2 - 2x_1^2}{2x_1y_1} = \frac{\left(\frac{y_1}{x_1}\right)^2 - 2}{2\left(\frac{y_1}{x_1}\right)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\left(\frac{y}{x}\right)^2 - 2}{2y/x}$$

Put $y = vx$

$$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx},$$

We have, $v + \frac{xdv}{dx} = \frac{v^2 - 2}{2v}$

$$\Rightarrow \quad x \cdot \frac{dv}{dx} = -\frac{(2 + v^2)}{2v}$$

$$\Rightarrow \quad \frac{2v \cdot dv}{2 + v^2} + \frac{dx}{x} = 0$$

On integrating both sides, we get

$$\log(2 + v^2) + \log|x| = \log C$$

$$\Rightarrow \quad \log(2 + v^2)|x| = \log C$$

$$\Rightarrow \quad |x|\left(2 + \frac{y^2}{x^2}\right) = C$$

$\because$ It passes through (2, 1) then, $2\left(2+\frac{1}{4}\right)=C$

$\Rightarrow \quad C=\frac{9}{2}$

$\therefore \quad \frac{|x|(2x^2+y^2)}{x^2}=\frac{9}{2}$

$\Rightarrow \quad 2x^2+y^2=\frac{9}{2}|x|$

$\Rightarrow \quad 4x^2+2y^2-9|x|=0$

52. $|\mathbf{a}\times\mathbf{b}|^2=|\mathbf{a}|^2|\mathbf{b}|^2-2(\mathbf{a}\cdot\mathbf{b})^2=16-4=12$ and

$|\mathbf{c}|^2=(2\mathbf{a}\times\mathbf{b}-3\mathbf{b})^2=(2\mathbf{a}\times\mathbf{b}-3\mathbf{b})$

$=4|\mathbf{a}\times\mathbf{b}|^2+9|\mathbf{b}|^2=4.12+9.16$

$=192\Rightarrow|\mathbf{c}|=8\sqrt{3}$

Now, $\quad \mathbf{b}\cdot\mathbf{c}=\mathbf{b}(2\mathbf{a}\times\mathbf{b}-3\mathbf{b})=-3|\mathbf{b}|^2=-48$

$\therefore \quad \cos\theta=\frac{\mathbf{b}\cdot\mathbf{c}}{|\mathbf{b}||\mathbf{c}|}=-\frac{48}{4\cdot 8\sqrt{3}}=-\frac{\sqrt{3}}{2}$

$\Rightarrow \quad \theta=\frac{5\pi}{6}$

53. Let the first term and common ratio of a GP be α and β, then

$a=\alpha\cdot\beta^{p-1}$, $b=\alpha\cdot\beta^{q-1}$ and $c=\alpha\cdot\beta^{r-1}$

$\therefore \quad \log a=\log\alpha+(p-1)\log\beta,$

$\log b=\log\alpha+(q-1)\log\beta$ and

$\log c=\log\alpha+(r-1)\log\beta$

The dot product of the given two vectors is

$(q-r)\log a+(r-p)\log b+(p-q)\log c$

$\Rightarrow(q-r)[\log\alpha+(p-1)\log\beta]+(r-p)$

$[\log\alpha+(q-1)\log\beta]+(p-q)[\log\alpha+(r-1)\log\beta]$

$\Rightarrow \quad \log\alpha\,[q-r+r-p+p-q]+$

$\log\beta\,[(p-1)(q-r)+(r-p)(q-1)$

$+(r-1)(p-q)]$

$=0+0=0$

$\Rightarrow$ The two vectors are perpedicular.

54. The straight line joining the points (1, 1, 2) and (3, –2, 1) is $\frac{x-1}{2}=\frac{y-1}{-3}=\frac{z-2}{-1}=r$ (say)

$\therefore$ point is $(2r+1, 1-3r, 2-r)$ which lies on

$3x+2y+z=6$

$\therefore \quad 3(2r+1)+2(1-3r)+2-r=6$

$\Rightarrow \quad r=1$

So, required point is (3, –2, 1).

55. Let Q be the point where the line through P (1, – 5, 9) and parallel to $\mathbf{r}=\hat{i}+\hat{j}+\hat{k}$ meets the plane $\mathbf{r}\,(\hat{i}-\hat{j}+\hat{k})=5$.

Equation of PQ, is

$\frac{x-1}{1}=\frac{y+5}{1}=\frac{z-9}{1}=\lambda$

$\therefore \quad x=\lambda+1,$

$y=\lambda-5$

and $\quad z=\lambda+9$

lies on the plane $x-y+z=5$

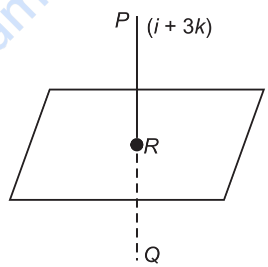
$\Rightarrow \quad \lambda+1-\lambda+5+\lambda+9=5$

$\Rightarrow \quad \lambda=-10$

$\therefore$ Co-ordinates of Q are Q (–9, – 15,–1)

$\Rightarrow \quad PQ=\sqrt{10^2+10^2+10^2}=10\sqrt{3}$

56. Let Q be the image of the point $P\,(\hat{i}+3\hat{k})$ in the plane $\mathbf{r}\cdot(\hat{i}+\hat{j}+\hat{k})=1$. Then PQ is normal to the plane. Since PQ passes through P and is normal to the given plane, therefore equation of PQ is



$\mathbf{r}=(\hat{i}+3\hat{k})+\lambda(\hat{i}+\hat{j}+\hat{k})$

Since, Q lies on the line PQ, so, let the position vector of Q be $(\hat{i}+3\hat{k})+\lambda(\hat{i}+\hat{j}+\hat{k})$

$\Rightarrow(1+\lambda)\hat{i}+\lambda\hat{j}+(3+\lambda)\hat{k}$

Since R is the mid point of PQ, therefore position vector of R is

$\frac{(1+\lambda)\mathrm{i}+\lambda\mathrm{j}+(3+\lambda)\hat{k}+\hat{i}+3\hat{k}}{2}$

or $\left(\frac{\lambda+2}{2}\right)\mathrm{i}+\left(\frac{\lambda}{2}\right)\hat{j}+\left(3+\frac{\lambda}{2}\right)\hat{k}$

or $\left(\frac{\lambda}{2}+1\right)\hat{i}+\left(\frac{\lambda}{2}\right)\hat{j}+\left(3+\frac{\lambda}{2}\right)\hat{k}$

Since, R less on the plane $\mathbf{r}\cdot(\hat{i}+\hat{j}+\hat{k})=1$

Therefore,
$$\left[\left(\frac{\lambda}{2}+1\right)\hat{i}+\left(\frac{\lambda}{2}\right)\hat{j}+\left(3+\frac{\lambda}{2}\right)\hat{k}\right]\cdot(\hat{i}+\hat{j}+\hat{k})=1$$
$$\Rightarrow \quad \left(\frac{\lambda}{2}+1+\frac{\lambda}{2}+3+\frac{\lambda}{2}\right)=1$$
$$\Rightarrow \quad \lambda=-2$$
So, the postion vector of Q is
$$(\hat{i}+3\hat{k})-2(\hat{i}+\hat{j}+\hat{k})=-\hat{i}-2\hat{j}+\hat{k}$$

57. Let the probability of success and failure be p and q, respectively

Then, $\quad p=2q$

and $\quad p+q=1$

$\Rightarrow \quad 3q-1$

$\Rightarrow \quad q=\frac{1}{3}$

$\therefore \quad p=\frac{2}{3}$

Required probability
$$={}^6C_4\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^2+{}^6C_5\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)+{}^6C_6\left(\frac{2}{3}\right)^6$$
$$=\frac{496}{729}$$

58. Since, m and n are selected between 1 and 100, hence sample space $=100\times 100$

Also, $\quad 7^1=1, 7^2=49, 7^3=343, 7^4=2401$

$7^5=16807$ etc.

Hence, 1, 3, 7 and 9 will be the last digits in the power of 7. Hence, for favourable cases.

nm
↓

1, 2   1, 2   1, 3   …   1, 100

2, 1   2, 2   2, 3   …   2, 100

100, 1   100, 2   100, 3   …   100, 100

for $m=1, \quad n=3, 7, 11, \quad \dots \quad 97$

$\therefore$ favourable cases $=25$

for $m=2, \quad n=4, 8, 12, 10 \quad \dots \quad 100$

$\therefore$ favourable cases $=25$

Similarly for every m, favourable n are 25.

$\therefore$ Total favourable cases $=100\times 25$

Hence, required probability $=\frac{100\times 25}{100\times 100}=\frac{1}{4}$

59. Let $\sin^{-1}a=A$
$$\sin^{-1}b=B$$
$$\sin^{-1}C=C$$
$$\therefore \quad \sin A=a, \quad \sin B=b, \quad \sin C=C$$
and $A+B+C=\pi$, then
$$\sin 2A+\sin 2B+\sin 2C$$
$$=4\sin A\sin B\sin C \quad \dots(i)$$
$$\Rightarrow \sin A\cos A+\sin B\cos B+\sin C\cos C$$
$$=2\sin A\sin B\sin C$$
$$\Rightarrow \sin A\sqrt{1-\sin^2 A}+\sin B\sqrt{1-\sin^2 B}$$
$$+\sin C\sqrt{1-\sin^2 C}=2\sin A\sin B\sin C \quad \dots(ii)$$
$$\Rightarrow a\sqrt{1-a^2}+b\sqrt{1-b^2}+c\sqrt{1-c^2}=2\,abc$$
while $\sin^{-1}a+\sin^{-1}b+\sin^{-1}c=\pi$

60. Given system of equation is
$$x+y+z=6$$
$$x+2y+3z=10$$
$$x+2y+\lambda z=\mu$$
If given system of equation has no solution, then $D=0$ and atleast one of the determinants $D_1, D_2, D_3\neq 0$

Here $\quad D=0$
$$\Rightarrow \quad \begin{vmatrix}1&1&1\\1&2&3\\1&2&\lambda\end{vmatrix}=0$$
$$\Rightarrow 1(2\lambda-6)-1(\lambda-3)+1(2-2)=0$$
$$\Rightarrow \quad 2\lambda-6-\lambda+3+0=0$$
$$\Rightarrow \quad \lambda-3=0$$
$$\Rightarrow \quad \lambda=3$$
Also, $D_1=\begin{vmatrix}6&1&1\\10&2&3\\\mu&2&3\end{vmatrix}\neq 0$
$$\Rightarrow 6(6-6)-1(30-3\mu)+1(20-2\mu)\neq 0$$
$$\Rightarrow \quad 0-30+3\mu+20-2\mu\neq 0$$
$$\Rightarrow \quad \mu-10\neq 0$$
$$\Rightarrow \quad \mu\neq 10$$

61. Given, $f(x)=sec\left(\frac{\pi}{4}\cos^2 x\right)$

We know that, $\quad 0\leq\cos^2 x\leq 1$
$$\therefore \quad f(x)=sec\left(\frac{\pi}{4}\times 1\right)=\sqrt{2}$$
$$f(x)=sec\left(\frac{\pi}{4}\times 0\right)=1 \quad \therefore \text{Range}\in[1, \sqrt{2}]$$

62. Let $y = \lim_{x \to 10} \left(\frac{a^x + b^x + c^x}{3}\right)^{2/x}$

$\Rightarrow \quad \log y = \lim_{x \to 10} \frac{2}{x} \log\left(\frac{a^x + b^x + c^x}{3}\right)$

$= \lim_{x \to 10} \frac{\log(a^x + b^x + c^x) - \log 3}{x}$

$\Rightarrow \quad \log y = \log (abc)^{2/3}$ (by L-Hospital's rule)

$\Rightarrow \quad y = (abc)^{2/3}$

63. We have $f(x) = [x] \cos\left[\frac{2x-1}{2}\right]\pi$

Since, $g(x) = [x]$ is always discontinuous at all integral values of points. Hence, $f(x)$ is discontinuous for all integral points.

64. we know that funtion $|x|$ is not differentiable at $x = 0$

$\therefore \quad |x^2 - 3x + 2| = |(x-1)(x-2)|$

Hence, it is not differentiable at $x = 1$ and 2

Now, $f(x) = (x^2 - 1)|x^2 - 3x + 2| + \cos|x|$ is not differentiable at $x = 2$.

For $1 < x < 2$, $f(x) = -(x^2 - 1)(x^2 - 3x + 2) + \cos x$

For $2 < x < 3$, $f(x) = (x^2 - 1)(x^2 - 3x + 2) + \cos x$

$L f'(x) = -(x^2 - 1)(2x - 3) - 2x(x^2 - 3x + 2) - \sin x$

$Lf'(2) = -3\sin 2$

$Rf'(x) = (x^2 - 1)(2x - 3) + 2x(x^2 - 3x + 2) - \sin x$

$Rf'(2) = (4-1)(4-3) + 0 - \sin 2 = 3 - \sin 2$

Hence, $Lf'(2) \neq Rf'(2)$.

So, $f(x)$ is not differentiable at $x = 2$.

65. Let $x = 5$, $y = 0 \Rightarrow f(5+0) = f(5) \cdot f(0)$

$\Rightarrow \quad f(5) = f(5)\,f(0) \Rightarrow f(0) = 1$

$\therefore \quad f'(5) \lim_{h \to 0} = \frac{f(5+h) - f(5)}{h}$

$= \lim_{h \to 0} \frac{f(5)\,f(h) - f(5)}{h}$

$= \lim_{h \to 0} 2\left(\frac{f(h) - 1}{h}\right) \quad [\because f(5) = 2]$

$= 2 \lim_{h \to 0} \left(\frac{f(h) - f(0)}{h}\right) = 2 \times f(0)$

$= 2 \times 3 = 6$

66. Here, $x \sin(a+y) + \sin a \cos(a+y) = 0 \quad \ldots$(i)

On differentiating w.r.t. x, we get

$\frac{d}{dx}[x \sin(a+y)] + \frac{d}{dx}[\sin a \cos(a+y)] = 0$

$\Rightarrow \left[x \frac{d}{dx}\{\sin(a+y)\} + \sin(a+y)\frac{d}{dx}(x)\right]$

$+ \sin a \frac{d}{dx} \cos(a+y) = 0$

(using product rule and chain rule)

$\Rightarrow \quad \left[x \cos(a+y) \frac{d}{dx}(a+y) + \sin(a+y)\right]$

$+ \sin a\left[-\sin(a+y)\frac{d}{dx}(a+y)\right] = 0$

$\Rightarrow \left[x \cos(a+y)\left(0 + \frac{dy}{dx}\right) + \sin(a+y)\right]$

$-\sin a \sin(a+y)\left(0 + \frac{dy}{dx}\right) = 0$

$\Rightarrow \quad x \cos(a+y)\frac{dy}{dx} + \sin(a+y)$

$-\sin a \sin(a+y)\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx}[x \cos(a+y) - \sin a \sin(a+y)]$

$= -\sin(a+y)$

$\left[\text{from Eq. (i), putting } x = -\sin a \frac{\cos(a+y)}{\sin(a+y)}\right]$

$\Rightarrow \quad \frac{dy}{dx}\left[-\sin a \frac{\cos^2(a+y)}{\sin(a+y)} - \sin a \sin(a+y)\right]$

$= -\sin(a+y)$

$\Rightarrow -\frac{dy}{dx}\left[\frac{\sin a \cos^2(a+y) + \sin a \sin^2(a+y)}{\sin(a+y)}\right]$

$= -\sin(a+y)$

$\Rightarrow \frac{dy}{dx} = \sin(a+y)$

$\left[\frac{\sin(a+y)}{\sin a\{\cos^2(a+y) + \sin^2(a+y)\}}\right]$

$\Rightarrow \quad \frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a} \qquad (\because \sin^2\theta + \cos^2\theta = 1)$

67. Firstly, break the number 5.001 as $x = 5$ and $\Delta x = 0.001$ and use the relation $f(x + \Delta x) \simeq f(x) + \Delta x\, f'(x)$.

Consider, $f(x) = x^3 - 7x^2 + 15$

$\Rightarrow \quad f'(x) = 3x^2 - 14x$

Let $\quad x = 5$

and $\quad \Delta x = 0.001$

Also, $f(x + \Delta x) \simeq f(x) + \Delta x\, f'(x)$

Therefore,

$f(x + \Delta x) \simeq (x^3 - 7x^2 + 15) + \Delta x\,(3x^2 - 14x)$

$$\Rightarrow \quad f(5.001) \simeq (5^3 - 7 \times 5^2 + 15) + (3 \times 5^2 - 14 \times 5)(0.001)$$

(as $x = 5$, $\Delta x = 0.001$)

$$= 125 - 175 + 15 + (75 - 70)(0.001)$$

$$= -35 + (5)(0.001)$$

$$= -35 + 0.005 = -34.995$$

68. Here, we observe that

(a) $f(x)$ is a polynomial, so it is continuous in the interval $[0, 2]$.

(b) $f'(x) = 3x^2 - 6x + 2$ exists for all $x \in (0, 2)$.

So, $f(x)$ is differentiable for all $x \in (0, 2)$ and

(c) $f(0) = 0$, $f(2) = 2^3 - 3(2)^2 + 2(2) = 0$

$\Rightarrow f(0) = f(2)$

Thus, all the three conditions of Rolle's theorem are satisfied.

So, there must exist $c \in [0, 2]$ such that $f'(c) = 0$

$\Rightarrow \quad f'(c) = 3c^2 - 6c + 2 = 0$

$\Rightarrow \quad c = 1 \pm \dfrac{1}{\sqrt{3}} \in [0, 2]$

69. The equation of any plane through the intersection of the planes,

$3x - y + 2z - 4 = 0$ and $x + y + z - 2 = 0$, is

$(3x - y + 2z - 4) + \lambda\,(x + y + z - 2) = 0 \quad \ldots(\text{i})$

The plane passes through the point (2, 2, 1). Therefore, this point will satisfy Eq. (i).

$\therefore\ (3 \times 2 - 2 + 2 \times 1 - 4) + \lambda\,(2 + 2 + 1 - 2) = 0$

$\Rightarrow \quad (6 - 4) + 3\lambda = 0$

$\Rightarrow \quad 2 + 3\lambda = 0$

$\Rightarrow \quad \lambda = \dfrac{-2}{3}$

On substituting this value of λ in Eq. (i), we get the required plane as

$$(3x - y + 2z - 4) - \frac{2}{3}(x + y + z - 2) = 0$$

$\Rightarrow \quad 9x - 3y + 6z - 12 - 2x - 2y - 2z + 4 = 0$

$\Rightarrow \quad 7x - 5y + 4z - 8 = 0$

This is the required equation of the plane.

70. Let $y = \operatorname{cosec}^2 x$,

Required limit $= \lim\limits_{y \to \infty} (1^y + 2^y + \ldots n^y)^{1/y}$

$(0/\infty$ form)

$$= \lim_{y \to \infty} (n^y)^{1/y} \left[\left(\frac{1}{n}\right)^y + \left(\frac{2}{n}\right)^y + \ldots + \left(\frac{n-1}{n}\right)^y + 1\right]^{1/y}$$

$$= \lim_{y \to \infty} n \left[\left(\frac{1}{n}\right)^y + \left(\frac{2}{n}\right)^y + \ldots + \left(\frac{n-1}{n}\right)^y + 1\right]^{1/y}$$

$= n \cdot 1^0 = n \cdot 1 = n$

71. Let $I = \displaystyle\int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cos x}\,dx \quad \ldots(\text{i})$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2} - x\right) - \cos\left(\frac{\pi}{2} - x\right)}{1 + \sin\left(\frac{\pi}{2} - x\right)\cos\left(\frac{\pi}{2} - x\right)}\,dx$$

$$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a - x)\,dx\right]$$

$$= \int_0^{\pi/2} \frac{\cos x - \sin x}{1 + \cos x \sin x}\,dx \quad \ldots(\text{ii})$$

$$\left[\because \sin\left(\frac{\pi}{2} - x\right) = \cos x \text{ and } \cos\left(\frac{\pi}{2} - x\right) = \sin x\right]$$

On adding Eqs. (i) and (ii), we get

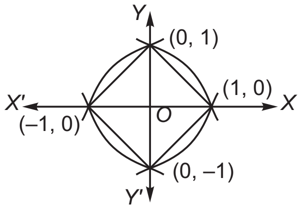
$$2I = \int_0^{\pi/2} \frac{0}{1 + \sin x \cos x}\,dx = 0 \quad \Rightarrow \quad I = 0$$

72. Since, $|x| + |y| = 1$

$$\Rightarrow \begin{cases} x + y = 1, & x > 0, \quad y > 0 \\ x - y = 1, & x > 0, \quad y < 0 \\ -x + y = 1, & x < 0, \quad y > 0 \\ -x - y = 1, & x < 0, \quad y < 0 \end{cases}$$

and $\quad 1 - y^2 = |x|$

$$\Rightarrow \begin{cases} 1-y^2 = x, & x \geq 0 \\ 1-y^2 = -x, & x < 0 \end{cases}$$



$\therefore$ Required area

$$= \left| 2\int_0^1 \sqrt{(1-x)}\,dx \right| + \left| 2\int_{-1}^0 \sqrt{(x+1)}\,dx \right| - 4\left(\frac{1}{2}\cdot 1\cdot 1\right)$$

$$= \frac{2}{3} \text{ sq unit}$$

73. Let slope of a line be m.

Now, the equation of a line passing through $(1, 0)$ is

$$y - 0 = m(x-1)$$

$$\Rightarrow \quad mx - y - m = 0$$

Distance from origin $= \dfrac{\sqrt{3}}{2}$

$$\Rightarrow \quad \frac{|-m|}{\sqrt{1+m^2}} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad 4m^2 = 3(1+m^2)$$

$$\Rightarrow \quad m^2 = 3$$

$$\Rightarrow \quad m = \pm\sqrt{3}$$

$\therefore$ Equations of lines are

$$\sqrt{3}x - y - \sqrt{3} = 0$$

and $$-\sqrt{3}x - y + \sqrt{3} = 0$$

$$\Rightarrow \quad \sqrt{3}x - y - \sqrt{3} = 0$$

and $$\sqrt{3}x + y - \sqrt{3} = 0$$

74. From figure, we have

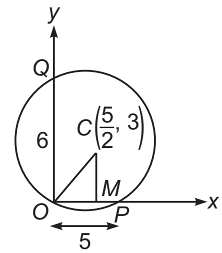
$$OP = 5, \quad OQ = 6$$

and $$OM = \frac{5}{2}, \quad CM = 3$$

$\therefore$ In $\Delta OMC, OC^2 = OM^2 + MC^2$

$$\Rightarrow \quad OC^2 = \left(\frac{5}{2}\right)^2 + (3)^2$$

$$\Rightarrow \quad OC = \frac{\sqrt{61}}{2}$$



Thus, the required circle has its centre $\left(\dfrac{5}{2}, 3\right)$ and radius $\dfrac{\sqrt{61}}{2}$.

So, its equation is $\left(x - \dfrac{5}{2}\right)^2 + (y-3)^2 = \left(\dfrac{61}{4}\right)$.

Hence, $\lambda = \dfrac{61}{4}$

75. Obtain the foot of the perpendicular $N(-4, 1, -3)$.

Find the equations of the line passing through $P(2, 4, -1)$ and $N(-4, 1, -3)$

$$\frac{x-2}{6} = \frac{y-4}{3} = \frac{z+1}{2}$$

76. Given, $\cos^2 A + \cos^2 C = \sin^2 B$

Obviously it is not an equilateral triangle because $A = B = C = 60°$ does not satisfy the given condition. But $B = 90°$, then $\sin^2 B = 1$ and

$$\cos^2 A + \cos^2 C = \cos^2 A + \cos^2\left(\frac{\pi}{2} - A\right)$$

$$= \cos^2 A + \sin^2 A = 1$$

Hence, this satisfies the condition, so it is a right angled triangle but not necessarily isosceles triangle.

77. $$\Delta = \begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix}$$

$$= \begin{vmatrix} 2(x+y+z) & x & y \\ 2(x+y+z) & y+z+2x & y \\ 2(x+y+z) & x & z+x+2y \end{vmatrix}$$

(using $C_1 \to C_1 + C_2 + C_3$)

Take out $2(x+y+z)$ common from C_1, we get

$$\Delta = 2(x+y+z)\begin{vmatrix} 1 & x & y \\ 1 & y+z+2x & y \\ 1 & x & z+x+2y \end{vmatrix}$$

$$= 2(x+y+z)\begin{vmatrix} 1 & x & y \\ 0 & y+z+x & 0 \\ 0 & 0 & z+x+y \end{vmatrix}$$

(using $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$)

Take out $(x+y+z)$ common from R_2 and R_3, we get

$$\Delta = 2(x+y+z)(x+y+z)(x+y+z) \times \begin{vmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding along R_3,we get

$$\Delta = 2(x+y+z)^3 [(1)(1-0)]$$

$$= 2(x+y+z)^3 = k(x+y+z)^3 \text{ (given)}$$

$\therefore \quad k = 2$

78. Let $\quad f(x) = \dfrac{x}{\log x}$

$$f'(x) = \frac{\log x - 1}{(\log x)^2}$$

Pur $f'(x) = 0$, for maxima or minima, we get

$$\log x - 1 = 0$$

$$\Rightarrow \quad x = e$$

Now, $f''(x) = \dfrac{(\log x)^2 \cdot \dfrac{1}{x} - (\log x - 1) \cdot \dfrac{2\log x}{x}}{(\log x)^4}$

$$\Rightarrow \quad f''(e) = \frac{\dfrac{1}{e} - 0}{1} = \frac{1}{e} > 0$$

$\therefore$ Function is minimum at $x = e$.

Hence, minimum value of $f(x)$ at $x = e$ is $f(e) = e$.

79. Total number of mappings from a set A having elements into itself is n^n.

And the total number of one to one mapping is $n!$.

$\therefore$ Required probability $= \dfrac{n!}{n^n}$

80. Total number of available courses = 9

Out of these 5 courses have to be chosen. But it is given that 2 courses are compulsory for every student *i.e.*, you have to choose only 3 courses instead of 5, out of 7 instead of 9.

It can be done in ${}^7C_3$ ways $= \dfrac{7 \times 6 \times 5}{6} = 35$ ways.