MET 2015 Answer Key PDF

Physics

1. Let the third charge $-q$ is placed at a distance of x from the charge 2q. Then, the potential energy of the system is

$$U = k\left[\frac{2q\,(8q)}{r} - \frac{(2q)\,(q)}{x} - \frac{(8q)\,(q)}{r-x}\right]$$

Here, $k = \frac{1}{4\pi\varepsilon_0}$

U is minimum, where $\frac{2}{x} + \frac{8}{r-x}$ is maximum

Let $\frac{2}{x} + \frac{8}{r-x} = y$

For y to be maximum, $\frac{dy}{dx} = 0$

$$-\frac{2}{x^2} + \frac{8}{(r-x)^2} = 0$$

$$\Rightarrow \quad \frac{x}{r-x} = \sqrt{\frac{2}{8}} = \frac{1}{2} \Rightarrow x = \frac{r}{3}$$

But at $x = \frac{r}{3}$, $\frac{d^2y}{dx^2}$ = positive i.e. at $x = \frac{r}{3}$, y is minimum or U is maximum. So, there cannot be any point between them where the potential energy is minimum.

2. At thermal equilibrium, the ratio $\frac{N_2}{N_1}$ is given as

$$\frac{N_2}{N_1} = \exp\left(-\frac{E_2 - E_1}{kT}\right)$$

The middle of the visible range is taken at $\lambda = 550$ nm

$$\Rightarrow \quad E_2 - E_1 = \frac{hc}{\lambda} = 3.16 \times 10^{-19}\,J$$

$$\Rightarrow \quad \frac{N_2}{N_1} = \exp\left(\frac{-3.16 \times 10^{-19}\,J}{(1.38 \times 10^{-23} 1/k)\cdot(300\ k)}\right)$$

$$\Rightarrow \quad \frac{N_2}{N_1} = 1.1577 \times 10^{-38}$$

3. The energy of the hydrogen atom in *n*th state is givne by

$$E_n = \frac{-13.6}{(n^2)}eV$$

Here, $n = 5$

$$\therefore \quad E = \frac{-13.6}{(5)^2}eV = -0.54\ eV$$

4. When a galvanometer is connected to the shunt resistance, then we have potential drop across the galvanometer = potential drop across the shunt

i.e. $\quad iG = (i - i_0)\,S \quad$ ...(i)

Here, i = current through the galvanometer

G = resistance of the galvanometer

i_0 = current through the shunt resistance

S = shunt resistance

From Eq. (i), we have the part of the total current that flows through the galvanometer is

$$\frac{i}{i_0} = \frac{S}{G+S}$$

Substituting, $S = 2.5\ \Omega$, $G = 25\ \Omega$, we get

$$\frac{i}{i_0} = \frac{1}{11}$$

5. $\because v_{rms} = \sqrt{\frac{3RT}{M}}$ and $v_{sound} = \sqrt{\frac{\gamma RT}{M}}$,

$$v_{rms} = 2\,v_{sound}$$

i.e. $\quad \gamma = \frac{3}{2}$ = ratio of $\frac{C_p}{C_V}$ for the mixture

$$C_V = \frac{n_1 C_{V_1} + n_2 C_{V_2}}{n_1 + n_2}$$

and $\quad C_p = \frac{n_1 C_{p_1} + n_2 C_{p_2}}{n_1 + n_2}$

$$\therefore \quad \gamma = \frac{C_P}{C_V} = \frac{n_1 C_{p_1} + n_2 C_{p_2}}{n_1 C_{V_1} + n_2 C_{V_2}}$$

$$\therefore \quad \frac{3}{2} = \frac{2\left(\frac{5}{2}R\right) + n\left(\frac{7}{2}R\right)}{2\left(\frac{3}{2}R\right) + n\left(\frac{5}{2}R\right)}$$

$$\Rightarrow \quad \frac{3}{2} = \frac{10 + 7n}{6 + 5n}$$

$$\Rightarrow \quad n = 2$$

6. Current is given by

$$I = \frac{E}{R'},$$

Substituting, $E = -\frac{nd\phi}{dt}$, we get

$$I = -\frac{n}{R'}\cdot\frac{d\phi}{dt} \quad ...(i)$$

Given that w_1 and w_2 are the values of the flux associated with one turn of the coil.

Therefore, Eq. (i) becomes,

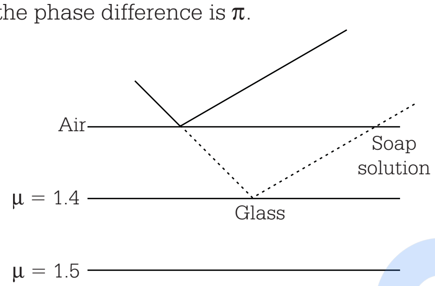
$$I = -\frac{n}{R'}\left[\frac{w_2 - w_1}{t_2 - t_1}\right] \quad \ldots\text{(ii)}$$

Total resistance of the combination is $R' = R + 4R = 5R$, substituting the values of R' and $(t_2 - t_1) = t$ in Eq. (ii), we get

$$I = -\frac{n}{5R}\left(\frac{w_2 - w_1}{t}\right)$$

$$\Rightarrow \quad I = \frac{-n(w_2 - w_1)}{5Rt}$$

7. For the reflection at the air-soap solution interface, the phase difference is π.



For reflection at the interface of soap solution to the glass also there will be a phase difference of π.

$\therefore$ The condition for maximum intensity $= 2\mu t = n\lambda$ ...(i)

For ,
$$n\lambda_1 = (n-1)\lambda_2$$
$$n(420) = (n-1)630$$
$$n(630 - 420) = 630$$
$$\therefore \quad n(210) = 630$$
$$\Rightarrow \quad n = 3$$

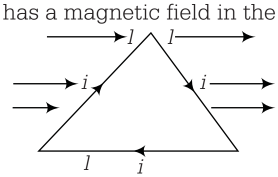
This is the maximum order where they coincide,

From Eq. (i),

$$\therefore \quad 2 \times 1.4 \times t = 3 \times 420$$

$$\Rightarrow \quad t = \frac{2 \times 420}{2 \times 1.40} = 450 \text{ nm}$$

8. The current flowing clockwise in an equilateral triangle has a magnetic field in the direction of $\hat{k}$.



$$\tau = BiNA \sin\theta$$
$$\tau = BiNA \sin 90^\circ$$
$$\tau = Bi \times \frac{\sqrt{3}}{4} l^2 \times 1$$

[$\because$ Area of equilateral triangle $= \frac{\sqrt{3}}{4} l^2$ and $N = 1$, (assuming]

$$\Rightarrow \quad l^2 = \frac{4\tau}{\sqrt{3}\,Bi} \Rightarrow l = 2\left[\frac{\tau}{Bi\sqrt{3}}\right]^{1/2}$$

9. Block of mass m_2 shoots off carrying some kinetic energy away from the system. To find its speed, potential energy of spring = maximum kinetic energy of blocks.

$$\frac{kd^2}{2} = (m_1 + m_2)\frac{v^2}{2}$$

[k = force constant of spring]

$$v^2 = \frac{kd^2}{m_1 + m_2} \text{ with } m_1 \text{ along on the spring.}$$

Maximum potential energy
= Maximum kinetic energy of m_2

$$\Rightarrow \quad \frac{1}{2}kA^2 = \frac{1}{2}m_1 v^2 \quad \Rightarrow kA^2 = \frac{km_1 d^2}{m_1 + m_2}$$

$$\Rightarrow \quad A = d\sqrt{\frac{m_1}{m_1 + m_2}}$$

10. Time taken by the small ball to return to the hands of the juggler is $\frac{2v}{g} = \frac{2 \times 20}{10} = 4$ s. So, he is throwing the balls after 1 s each. Let at some instant, he throws the ball number 4. Before 1 s of throwing it, he throws ball 3. So, the height of ball 3 is

$$h = ut - \frac{1}{2}gt^2$$

$$h_3 = 20 \times 1 - \frac{1}{2}(10)(1)^2 = 15 \text{ m}$$

Before 2 s, he throws ball 2. So, the height of ball 2 is

$$h_2 = 20 \times 2 - \frac{1}{2} \times 10\,(2)^2 = 20 \text{ m}$$

Before 3 s, he throws ball 1. So, the height of ball 1 is

$$h_1 = 20 \times 3 - \frac{1}{2} \times 10\,(3)^2$$

$$\Rightarrow \quad h_1 = 15 \text{ m}$$

11. The instantaneous current in the AC circuit is given by $I = I_0 \sin \omega t$

$$\frac{dI}{dt} = \frac{d}{dt}(I_0 \sin \omega t)$$

$$\frac{dI}{dt} = (I_0 \cos\omega t)\,\omega$$

Substituting $I_0 = 10$ A, $\omega = 100\pi$

$\cos(100\pi) = 1$, we get

$$\frac{dI}{dt} = 1000\pi$$

Hence, induced emf is given by

$$E = M\frac{dI}{dt}$$

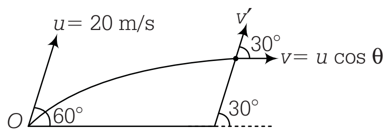
On putting $M = 0.005$ H, $\frac{dI}{dt} = 1000\pi$, we get

$$E = (5\pi)\ V$$

12. Speed of ball before collision is

$$v = u\cos 60° = 20 \times \frac{1}{2} = 10\ m/s$$

Since, collision is perfectly inelastic ($e = 0$), so the ball will not bounce. It will move along the plane with velocity



$$v' = v\cos 30° = \frac{10\sqrt{3}}{2} = 5\sqrt{3}\ m/s$$

$\therefore$ Maximum height attained by the ball

$$H = \frac{u^2\sin^2 60°}{2g} + \frac{v'^2}{2g}$$

$$H = \frac{(20)^2\left(\frac{\sqrt{3}}{2}\right)^2}{20} + \frac{(5\sqrt{3})^2}{20}$$

$$= \frac{300}{20} + \frac{75}{20} = 18.75\ m$$

13. Total energy produced by the reactor in time $t = 10$ yr.

$$E = 1000 \times 10^6 \times 10 \times 3.15 \times 10^7\ J$$

$$= 3.15 \times 10^{17}\ J$$

$\because$ Efficiency $= \frac{\text{output energy}}{\text{input energy}}$

Input energy caused by fission $= \frac{\text{output energy}}{\text{efficiency}}$

$$= \frac{3.15 \times 10^{17}}{(10/100)} = 3.15 \times 10^{18}\ J$$

Energy produced by one fission of $^{235}U = 200$ MeV

$$= 200 \times 1.6 \times 10^{-13}\ J$$

$$= 3.2 \times 10^{-11}\ J$$

Therefore, number of fissions required

$$= \frac{\text{total energy}}{\text{energy per fission}}$$

$$= \frac{3.15 \times 10^{18}}{3.2 \times 10^{-11}} = 9.8 \times 10^{28}$$

Hence, mass of uranium required is given by

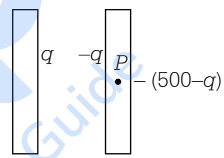
$$m = \frac{N}{N_a} \times 235 = \frac{9.8 \times 10^{28}}{6.02 \times 10^{26}}$$

$$= 38.2 \times 10^3\ kg$$

14. Charge acquired by the plates of the capacitor $q_0 = CV = (10\ \mu F) \times (50 V) = 500\ \mu C$

Now, let the charge distribution is as follows.

Total charge on positive plate has now become $700\ \mu C$ while that in negative plate is still $-500\ \mu C$,



Here, charges are in μC.

Net electric field at point P is zero.

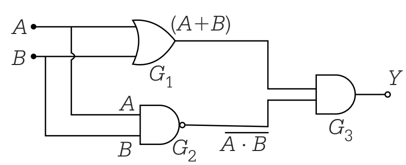
$$\therefore \quad \frac{(700 - q)}{2A\varepsilon_0} + \frac{q}{2A\varepsilon_0} + \left(\frac{500 - q}{2A\varepsilon_0}\right) = \frac{q}{2A\varepsilon_0}$$

$$\Rightarrow \quad q = 600\ \mu C$$

$\therefore$ Potential difference between the plates is

$$\Delta V = \frac{q}{C} = \frac{600\ \mu C}{10\ \mu F} = 60\ V$$

15.



$$Y = (A + B)\cdot\overline{AB}$$

The given output equation can also be written as

$Y = (A + B)\cdot(\overline{A} + \overline{B})$ (Demorgan's theorem)

$$\Rightarrow \quad Y = A\overline{A} + A\overline{B} + B\overline{A} + B\overline{B}$$

$$\Rightarrow \quad Y = 0 + A\overline{B} + \overline{A}B + 0$$

$$\Rightarrow \quad Y = \overline{A}B + A\overline{B}$$

This is the expression for XOR gate.

16. The current through the circuit before the battery of emf E_2 is short circuited, is

$$i_1 = \frac{E_1 + E_2}{r_1 + r_2 + R} \quad \ldots(i)$$

After short circuiting the battery of emf E_2, current through the resistance R would be

$$i_2 = \frac{E_1}{R + r_1} \quad \ldots(ii)$$

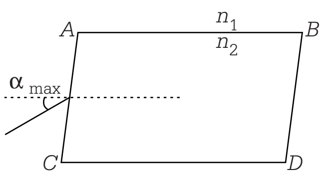
Since, from Eqs. (i) and (ii), we get

$$i_2 > i_1$$

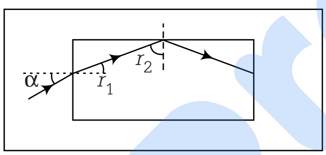
$$\therefore \quad \frac{E_1}{R + r_1} > \frac{E_1 + E_2}{R + r_1 + r_2}$$

This gives $\quad E_1 r_2 > E_2 (R + r_1)$

17.



Rays comes out only from CD, means rays after refraction from AB get totally internally reflected at AD.



From the figure,

$$r_1 + r_2 = 90$$

$$r_1 = 90 - r_2$$

$$(r_1)_{max} = 90° - (r_2)_{max} \text{ and } (r_2)_{min} = \theta_C$$

(for total internal reflection at AD)

where, $\quad \sin\theta_C = \frac{n_2}{n_1}$ or $\theta_C = \sin^{-1}\left(\frac{n_2}{n_1}\right)$

$$\therefore \quad (r_1)_{max} = 90° - \theta_C$$

Now, applying Snell's law at face AB, we wet

$$\frac{n_1}{n_2} = \frac{\sin\alpha_{max}}{\sin(r_1)_{max}} = \frac{\sin\alpha_{max}}{\sin(90 - \theta_C)} = \frac{\sin\alpha_{max}}{\cos\theta_C}$$

$$\sin\alpha_{max} = \frac{n_1}{n_2}\cos\theta_C$$

$$\therefore \quad \alpha_{max} = \sin^{-1}\left[\frac{n_1}{n_2}\cos\theta_C\right]$$

$$= \sin^{-1}\left[\frac{n_1}{n_2}\cos\left(\sin^{-1}\left(\frac{n_2}{n_1}\right)\right)\right]$$

18. We start a general form for a rightward moving wave,

$$y(x,t) = A\sin(kx - \omega t + \phi) \quad \ldots(i)$$

The given amplitude is $A = 2\text{ cm} = 0.02\text{ m}$

The wavelength is given as

$$\lambda = 1\text{ m}$$

Wave number $= k = 2\pi/\lambda = 2\pi\,\text{m}^{-1}$

Angular frequency,

$$\omega = \nu k = 10\pi\text{ rad/s}$$

From Eq. (i),

$$y(x,t) = (0.02)\sin[2\pi(x - 5t) + \phi]$$

$\because \quad$ For $x = 0$, $t = 0$

$$y = 0 \text{ and } \frac{\partial y}{\partial t} < 0$$

i.e. $\quad 0.02\sin\phi = 0 \quad$ (as $y = 0$)

and $\quad -0.2\pi\cos\phi < 0$

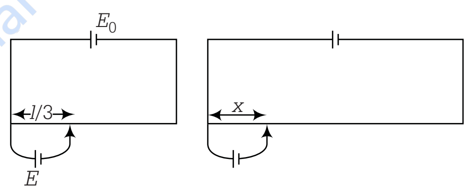
From these conditions, we may conclude that

$$\phi = 2n\pi, \text{ where } n = 0, 2, 4, 6, \ldots$$

Therefore,

$$y(x,t) = (0.02\text{ m})\sin[(2\pi\text{m}^{-1})x - (10\pi\text{s}^{-1})t]\text{ m}$$

19. Let x be the desired length.



Potential gradient in the first case $= \frac{E_0}{l}$

$$E = \left(\frac{l}{3}\right)\cdot\left(\frac{E_0}{l}\right) = \frac{E_0}{3} \quad \ldots(i)$$

Potential gradient in the second case.

$$= \frac{E_0}{3l/2} = \frac{2E_0}{3l}$$

$$\therefore \quad E = (x)\frac{2E_0}{3l} \quad \ldots(ii)$$

From Eqs. (i) and (ii), we get

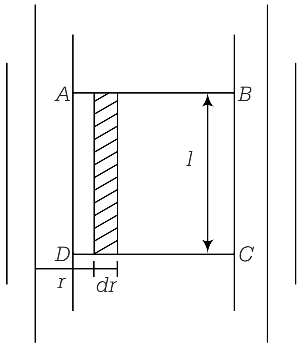
$$\frac{E_0}{3} = \left(\frac{2E_0}{3l}\right)x$$

$$\Rightarrow \quad x = \frac{l}{2}$$

20. Since, the wires are infinite, so the system of these two wires can be considered as closed rectangle of infinite length and breadth equal to d.

Flux through strip

$$\phi = \int_a^{d-a} \frac{\mu_0 I}{2\pi r}(l dr) = \frac{\mu_0 I l}{2\pi} \ln\left(\frac{d-a}{a}\right)$$



The other wire produces the same result, so the total flux through the rectangle ABCD is

$$\phi_{total} = \frac{\mu_0 I l}{\pi} \ln\left(\frac{d-a}{a}\right)$$

The total inductance of length l

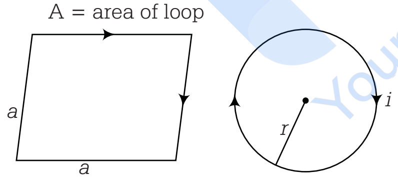
$$L = \frac{l_{total}}{I} = \frac{\mu_0 l}{\pi} \ln\left(\frac{d-a}{a}\right)$$

Inductance per unit length $= \frac{L}{l} = \frac{\mu_0}{\pi} \ln\left(\frac{d-a}{a}\right)$

21. Magnetic moment of a loop is given by $p_m = IA$

where, I = current through the loop

A = area of loop



(i) (ii)

Perimeter of square should be equal to the circumference of circle, since wires have same length.

Therefore, $4a = 2\pi r \Rightarrow a = \frac{\pi r}{2}$ ...(i)

In Fig. (i), magnetic moment of square loop is given by

$$p_{m_1} = I(A_1) \Rightarrow$$

$$\Rightarrow \quad p_{m_1} = \frac{I\pi^2 r^2}{4} \quad \text{[from Eq. (i)]} \ldots\text{(ii)}$$

In Fig. (ii), magnetic moment of circular loop is given by

$$p_{m_2} = I(A_2)$$

$$p_{m_2} = I(\pi r^2) \quad \ldots\text{(iii)}$$

From Eqs. (ii) and (iii), we get

$$\Rightarrow \quad \frac{p_{m_1}}{p_{m_2}} = \frac{\pi}{4}$$

22. The potential energy of circular loop in the magnetic field $= -MB\cos\theta$, where θ is the angle between normal to plane of coil and magnetic field B.

Initial potential energy,

$$U_i = -NiAB\cos 0° = -NiAB$$

When coil is turned through 180°, therefore final potential energy,

$$U_f = -NiAB\cos 180° = NiAB$$

$\therefore$ Required work, W = gain in potential energy

$$= U_f - U_i = NiAB - (-NiAB) = 2NiAB$$

Here, $N = 100$, $i = 0.1$ A, $B = 1.5$ Wb/m^2 and radius $r = 0.05$ m

$$\therefore \quad A = \pi r^2 = 3.14 \times (0.05)^2 = 7.85 \times 10^{-3} \text{ m}^2$$

$\therefore$ Work, $W = 2 \times 100 \times 0.1 \times 7.85 \times 10^{-3} \times 1.5$

$\Rightarrow \quad W = 0.2355$ J

23. The instantaneous current in L-R circuit during growth is given by

$$I = I_0\left(1 - e^{-\frac{R}{L}t}\right)$$

Since, all the exponential terms are dimensionless, therefore, it follows that factor $\frac{R}{L}t$ should be dimensionless i.e.

$$\frac{R}{L}t = [M^0 L^0 T^0]$$

$$\Rightarrow \quad \frac{R}{L} = \frac{[M^0 L^0 T^0]}{[T]} = [T^{-1}]$$

$$\Rightarrow \quad \left[\frac{L}{R}\right] = [T]$$

Hence, the time constant of L-R circuit is $\left[\frac{L}{R}\right]$.

24. In wave mechanics, the magnitude of angular momentum of an electron is

$$L = \sqrt{l(l+1)}\frac{h}{2\pi}$$

where, l can have the values 0, 1, 2......(n + 1)

n is principal quantum number.

25. The general equations of alternating voltage and current in an AC circuit are as follows

$$E = E_0 \sin \omega t \qquad \ldots\text{(i)}$$

$$I = I_0 \sin \omega t \qquad \ldots\text{(ii)}$$

Given that,

$$E = 100 \sin 100\pi t \qquad \ldots\text{(iii)}$$

$$I = 5 \sin 100\ \pi t \qquad \ldots\text{(iv)}$$

Comparing Eqs. (ii) and and (iv), we get,

Peak value of current, $I_0 = 5\,A$

Comparing Eqs. (i) and (iii), we get

Peak value of voltage, $E_0 = 100$ V

Now, r.m.s. value of voltage is given by

$$E_{rms} = \frac{E_0}{\sqrt{2}} = \frac{100}{\sqrt{2}}\ V$$

Similarly, r.m.s. value of current is given by

$$I_{rms} = \frac{I_0}{\sqrt{2}} = \frac{5}{\sqrt{2}}\ A$$

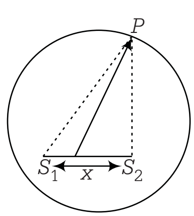
Hence, reactance of the circuit is given by

$$Z = \frac{E_{rms}}{I_{rms}} = \frac{\frac{100}{\sqrt{2}}}{\frac{5}{\sqrt{2}}} = 20\ \Omega$$

26. The damage caused by the γ-radiation is less serious as compared to damage caused by the α-radiation. This is because α-radiation is more dangerous than beta or gamma if ingested because its power to ionize or to disrupt, atoms is 20 times more than β and γ. But if the source is outside the body or at a distance gamma radiation is much dangerous because it could penertrate thick walls of skin.

27. Path difference at P is

$$\Delta x = 2\left(\frac{x}{2}\cos\theta\right) = x\cos\theta$$



For intensity to be maximum

$$\Delta x = n\lambda \qquad (n = 0, 1, \ldots\ldots)$$

$$\Rightarrow \qquad x\cos\theta = n\lambda$$

$$\cos\theta = \frac{n\lambda}{x} \text{ or } \cos\theta \ngtr 1$$

$$\therefore \qquad \frac{n\lambda}{x} \ngtr 1$$

$$\therefore \qquad n \ngtr \frac{x}{\lambda}$$

Substituting $x = 5\lambda$

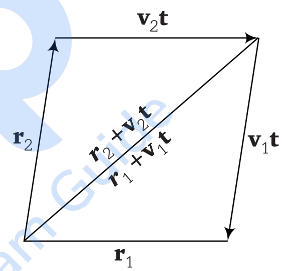
$$n \ngtr 5 \text{ or } n = 1, 2, 3, 4, 5, \ldots\ldots$$

Therefore, in all four quadrants, there can be 20 maximas. There are more maximas at $\theta = 0°$ and $\theta = 180°$.

But $n = 5$ corresponds to $\theta = 90°$ and $\theta = 270°$, which are coming only twice. While we have multiplied it four times. Therefore, total number of maximas still 20.

28. For collision, $\mathbf{r}_1 + \mathbf{v}_1 t = \mathbf{r}_2 + \mathbf{v}_2 t$

$$\Rightarrow \qquad \mathbf{r}_1 - \mathbf{r}_2 = (\mathbf{v}_2 - \mathbf{v}_1)t$$



Equating unit vectors, we get

$$\frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_2 - \mathbf{r}_1|} = \frac{\mathbf{v}_1 - \mathbf{v}_2}{|\mathbf{v}_2 - \mathbf{v}_1|}$$

29. Let v = speed of neutron before collision

v_1 = speed of neutron after collision

v_2 = speed of hydrogen atom after collision and ΔE = energy of the excitation

From the conservation of linear momentum,

$$mv = mv_1 + mv_2 \qquad \ldots\text{(i)}$$

From the conservation of energy,

$$\frac{1}{2}mv^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2 + \Delta E \qquad \ldots\text{(ii)}$$

from Eq. (i),

$$v^2 = v_1^2 + v_2^2 + 2v_1v_2$$

From Eq. (ii),

$$v^2 = v_1^2 + v_2^2 + \frac{2\Delta E}{m}$$

$$\therefore \qquad 2v_1v_2 = \frac{2\Delta E}{m}$$

$$\therefore \qquad (v_1 - v_2)^2 = (v_1 + v_2)^2 - 4v_1v_2$$

$$= v^2 - \frac{4\Delta E}{m}$$

As, $v_1 - v_2$ must be real

$\Rightarrow \quad v^2 - \dfrac{4\Delta E}{m} \geq 0$

or $\quad \dfrac{1}{2} mv^2 \geq 2\Delta E$

The maximum energy that can be absorbed by hydrogen atom in ground state to go into excited state is 10.2 eV. Therefore, the minimum kinetic energy required is

$$\frac{1}{2} mv_{min}^2 = 2 \times 10.2 \text{ eV} = 20.4 \text{ eV}$$

30. Molar specific heat

= Molecular weight × gram specific heat.

$$C_V = M \times c_V$$

$$2.98 \text{ cal/mol-K} = M \times 0.075 \text{ kcal/kg-K}$$

$$= \frac{M \times 0.075 \times 10^3}{10^3} \text{ cal/g-K}$$

∴ Molecular weight of argon,

$$M = \frac{2.98}{0.075} = 39.7 \text{ g}$$

i.e. mass of 6.023×10^{23} atom $= 39.7$ g

Therefore, mass of single atom $= \dfrac{39.7}{6.023 \times 10^{23}}$

$$= 6.60 \times 10^{-23} \text{g}$$

31. Radius of a nucleus is given by

$$R = R_0 A^{1/3} \text{ (where, } R_0 = 1.25 \times 10^{-15} \text{ m)}$$

$$= 1.25 \times 10^{-15} A^{1/3} \text{m}$$

Here, A is the mass number and mass of the uranium nucleus will be

$$m \approx Am_p, \; m_p = \text{mass of proton}$$

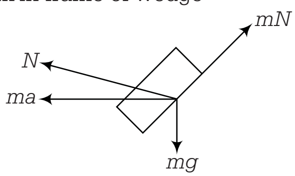
$$= A \,(1.67 \times 10^{-27} \text{ kg})$$

$$\therefore \quad \text{Density} = \frac{\text{Mass}}{\text{Volume}} = \frac{m}{\frac{4}{3}\pi R^3}$$

$$= \frac{A\,(1.67 \times 10^{-27})}{A \times \frac{4}{3} \times \frac{22}{7} \times (1.25 \times 10^{-15})^3}$$

$$\Rightarrow \quad \rho = 2 \times 10^{17} \text{ kg/m}^3$$

32. FBD of m in frame of wedge



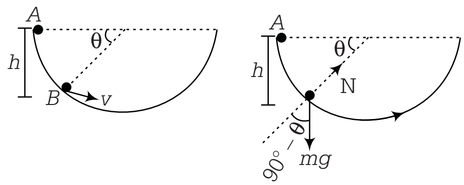
$$N = mg \cos(\alpha) - ma \sin(\alpha)$$

Now, $f = \mu N = ma \cos\alpha + mg \sin\alpha$

$$\Rightarrow \quad \mu = \frac{a \cos\alpha + g \sin\alpha}{g \cos\alpha - a \sin\alpha}$$

$$\Rightarrow \quad \mu = \frac{a + g \tan\alpha}{g - a \tan\alpha} = \frac{5}{12} \Rightarrow \mu = \frac{5}{12}$$

33. $\because \quad h = R \sin\theta$



∴ Speed of the particle at point B,

$$v^2 = 2gh = 2Rg \sin\theta$$

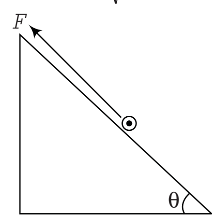
Centripetal force, $F = \dfrac{mv^2}{R} = 2mg \sin\theta \qquad \ldots(i)$

Equation of motion gives

$$N - mg \sin\theta = \frac{mv^2}{R} = 2\, mg\sin\theta$$

34. ∵ Linear acceleration for rolling,

$$a = \frac{g \sin\theta}{\sqrt{1 + k^2/R^2}}$$



For cylinder, $\dfrac{k^2}{R^2} = \dfrac{1}{2}$

$$\therefore \quad a_{cylinder} = \frac{2}{3} g \sin\theta$$

For rotation, the torque

$$FR = I\alpha \cdot (MR^2\alpha)/2$$

(where, F = force of friction)

But $\quad R\alpha = a \quad \therefore \quad F = \dfrac{M}{2} a$

$$F = \frac{M}{2} \cdot \frac{2}{3} g \sin\theta = \frac{M}{3} g \sin\theta$$

$\therefore \quad \mu_s = \dfrac{F}{N}$, where N is normal reaction,

$$\mu_s = \frac{\frac{M}{3} g \sin\theta}{Mg \cos\theta} = \frac{\tan\theta}{3}$$

∴ For rolling without slipping of a roller down, the inclined plane, $\tan\theta \leq 3\mu_s$.

35. Applying Newton's second law to a circular orbit, we have

$$\frac{mv^2}{r} = \frac{4\pi^2 rm}{T^2} = \frac{GMm}{r^2}$$

where, m is the mass of satellite, and v is the orbital speed

T is the time period

$$\therefore \quad T = \frac{2\pi r^{3/2}}{\sqrt{GM}}$$

For $\quad r \approx R$

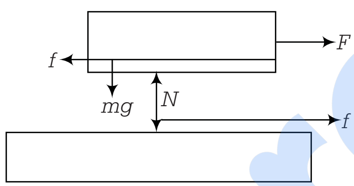
and $\quad M = \frac{4}{3}\pi R^3 \rho$ (ρ = density of planet)

$$\because \quad T = \sqrt{\frac{3\pi}{\rho G}}$$

i.e. is independent of R

36. Maximum frictional force between the slab and the block

$$f_{max} = \mu N = \mu mg = \frac{1}{4} \times 4 \times 10 = 10 \text{ N}$$



Evidently, $f \propto f_{max}$

So, the two bodies will move together as a single unit. If be their combined acceleration, then

$$a = \frac{F}{m + M} = \frac{6}{4 + 5} = \frac{2}{3} \text{ ms}^{-2}$$

Therefore, frictional force acting can be obtained as

$$f = Ma = \frac{2}{3} \times 5 = \frac{10}{3} \text{N}$$

Using $\quad S = \frac{1}{2}at^2$

$$S(2) = \frac{1}{2} \times \frac{2}{3}(2)^2 = \frac{4}{3}$$

and $\quad S(3) = \frac{1}{2} \times \frac{2}{3} \times (3)^2 = 3$

Therefore, work done by friction = F [S (3) – S (2)]

$$= \frac{10}{3}\left[3 - \frac{4}{3}\right] = \frac{50}{9} = 5.55 \text{ J}$$

37. Net pulling force = 2g – 1g = 10 N

Mass being pulled = 2 + 1 = 3 kg

$\therefore$ Acceleration of the system is $a = \frac{10}{3}$ m/s^2

$\therefore$ Velocity of both the blocks at t = 1 s will be

$$v_0 = at = \left(\frac{10}{3}\right)(1) = \frac{10}{3} \text{ m/s}$$

Now at this moment, velocity of 2 kg becomes zero, while that of 1 kg block is 10 m/s upwards. Hence, string becomes tight again when displacement of 1 kg block = displacement of 2 kg block.

$$\Rightarrow \quad v_0 t - \frac{1}{2}gt^2 = \frac{1}{2}gt^2$$

$$\Rightarrow \quad t = \frac{v_0}{g} = \frac{\frac{10}{3}}{10} = \frac{1}{3} \text{ s}$$

38. *(c)* From Snell's law,

$$\mu_1 \sin i = \mu_2 \sin r$$

$$\Rightarrow \quad \sin r = \frac{\mu_1}{\mu_2}\sin i = \frac{1}{\sqrt{2}}\sin 45°$$

$$\Rightarrow \quad \frac{1}{2} = \sin r \Rightarrow r = 30°$$

This means that the ray becomes parallel to side AD inside the slab.

This implies for the second face CD, u = ∞.

Given $\quad R = 0.4$ m

$$\frac{\mu_3}{v} - \frac{\mu_2}{v} = \frac{\mu_3 - \mu_2}{R}$$

$$\Rightarrow \quad \frac{1.514}{v} - \frac{\sqrt{2}}{\infty} = \frac{1.514 - \sqrt{2}}{R}$$

$$v = \frac{1.514 \times 0.4}{1.514 - 1.414} = \frac{1.514 \times 0.4}{0.1}$$

v = 6.056 = 6.06 m (upto 2 decimal places)

39. Energy required to raise a satellite upto a height h,

$$E_1 = \Delta U = \frac{mgh}{1 + \frac{h}{R}} \quad \ldots\text{(i)}$$

E_2 = energy required to put in orbit

$$= \frac{1}{2}mv_0^2$$

$$= \frac{1}{2}m\left(\frac{GM}{r}\right) \text{ as } v_0 = \text{orbital speed}$$

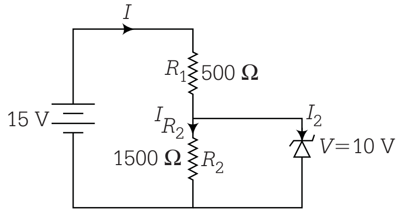
$$= \frac{1}{2}m\left(\frac{GM}{R + h}\right) = \frac{1}{2}m\left(\frac{GM}{R^2}\right)\frac{R}{1 + \frac{h}{R}}$$

$$\Rightarrow \quad E_2 = \frac{mgR}{2\left(1 + \frac{h}{R}\right)} \quad \ldots\text{(ii)}$$

From Eqs. (i) and (ii), we get

$$\frac{E_1}{E_2} = \frac{2h}{R}$$

40.



The voltage drop across R_2 is

$$V_{R_2} = V_z = 10\text{ V}$$

The current through R_2 is

$$I_{R_2} = \frac{V_{R_2}}{R_2} = \frac{10}{1500} = 0.667 \times 10^{-2}\text{ A}$$

$$= 6.67 \times 10^{-3}\text{ A} = 6.67\text{ mA}$$

The voltage drop across R_1 is

$$V_{R_1} = 15V - V_{R_2} = 15V - 10\text{ V} = 5\text{ V}$$

The current through R_1 is

$$I_{R_1} = \frac{V_{R_1}}{R_1} = \frac{5V}{500\ \Omega} = 10^{-2}\text{ A} = 10 \times 10^{-3} = 10\text{ mA}$$

The current through Zener diode is

$$I_z = I_{R_1} - I_{R_2} = (10 - 6.67)\text{ mA} = 3.33\text{ mA}$$

41. $\because f \propto \sqrt{T}$

$$\frac{f_{air}}{f_{water}} = \sqrt{\frac{W_{air}}{W_{water}}} = \sqrt{\frac{V\rho g}{V\rho g - V\rho_w g}}$$

$$= \frac{f}{\frac{f}{2}} = \sqrt{\frac{\rho}{\rho - \rho_w}}$$

$$\Rightarrow \quad 2 = \sqrt{\frac{\rho}{\rho - \rho_w}}$$

$$\Rightarrow \quad 4\rho - 4\rho_w = \rho$$

$$\Rightarrow \quad \rho = \frac{4}{3}\rho_w$$

Similarly in second case,

$$\frac{f}{\frac{f}{3}} = \sqrt{\frac{\rho}{\rho - \rho_L}}$$

$$\Rightarrow \quad 3 = \sqrt{\frac{\frac{4}{3}\rho_w}{4 - \frac{3\rho_L}{\rho_w}}} = \sqrt{\frac{4}{4 - \frac{3\rho_L}{\rho_w}}}$$

Here, $\frac{\rho_L}{\rho_w}$ = specific gravity (say S)

$$\Rightarrow \quad 9 = \frac{4}{4 - 3S}$$

$$\Rightarrow \quad 36 - 27S = 4 \Rightarrow S = \frac{32}{27}$$

42. According to Einstein's photoelectric equation,

$$E = h\nu - w$$

If V_s is retarding or stopping potential and ν_0 is the threshold frequency, then the above equation becomes

$$eV_s = h\nu - h\nu_0$$

$$h\nu = eV_s + h\nu_0$$

$$\nu = \frac{eV_s}{h} + \nu_0$$

Hence, $e = 1.6 \times 10^{-19}$ C

$$V_s = 3\text{ V},\ \nu_0 = 6 \times 10^{-14}\text{ Hz}$$

Therefore, required frequency,

$$\nu = \frac{1.6 \times 10^{-19} \times 3}{6.63 \times 10^{-34}} + 6 \times 10^{14}$$

$$= \frac{4.8}{6.63} \times 10^{15} + 6 \times 10^{14}$$

$$= 0.723 \times 10^{15} + 6 \times 10^{14}$$

$$= 7.23 \times 10^{14} + 6 \times 10^{14}$$

$$= 13.23 \times 10^{14}\text{ Hz}$$

$$= 1.323 \times 10^{15}\text{ Hz}$$

43. At $x_1 = \frac{\pi}{3k}$ and $x_2 = \frac{3\pi}{2k}$

$\sin kx_1$ or $\sin kx_2$ is not zero.

Therefore, neither of x_1 nor x_2 is a node.

$$\Delta x = x_2 - x_1 = \left(\frac{3}{2} - \frac{1}{3}\right)\frac{\pi}{k} = \frac{7\pi}{6k}$$

Since, $\frac{2\pi}{k} > \Delta x > \frac{\pi}{k}$

$$\lambda > \Delta x > \frac{\lambda}{2} \qquad \left\{k = \frac{2\pi}{\lambda}\right\}$$

Therefore, $\phi_1 = \pi$

and $\phi_2 = k$

$$\Delta x = \frac{7\pi}{6}$$

Therefore, $\frac{\phi_1}{\phi_2} = \frac{6}{7}$

44. Energy is given by

$$E = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\Rightarrow \quad E^2 = \frac{m_0^2 c^6}{c^2 - v^2}$$

Momentum P is given by

$$P = \frac{m_0 v}{\sqrt{1 - \left(\frac{v^2}{c^2}\right)}}$$

$$P^2c^2 = \frac{m_0^2c^4v^2}{c^2 - v^2}$$

$\therefore \quad E^2 - P^2c^2 = m_0^2c^4$

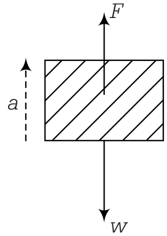
$\Rightarrow \quad E^2 = P^2c^2 + m_0^2c^4$

For photon, rest mass

$m_0 = 0$, so $E = Pc$

For electron, $m_0 \neq 0$, so $E \neq Pc$

45. Let m be the mass of block and its acceleration in upward direction. Then,



$$a = \frac{\text{upward thrust} - \text{weight}}{m} = \frac{F - w}{m}$$

$$\Rightarrow \quad a = \frac{\left(\frac{m}{0.5 \times 10^3}\right) \times (10^3) \times (g + a_0) - mg}{m}$$

$$\Rightarrow \quad a = 2\left(g + \frac{g}{2}\right) - g = 2g$$

Acceleration of block relative to water

$$a_r = a - a_0 = 2g - \frac{g}{2} = \frac{3g}{2}$$

$$\therefore \quad h = \left(\frac{1}{2}\right) a_r t^2$$

$$\Rightarrow \quad t = \sqrt{\frac{2h}{a_r}} = \sqrt{\frac{2h}{\frac{3}{2}g}} = 2\sqrt{\frac{h}{3g}} = 2\sqrt{\frac{1.2}{3 \times 10}}\ \text{m}$$

$$\Rightarrow \quad t = 0.4\ \text{s}$$

46. Energy $= \frac{1}{2}mv^2 = 5000$ eV

$= 5000 \times 1.6 \times 10^{-19}$ J

$mv_2 = \sqrt{2 \times 5000 \times (1.6 \times 10^{-19})\ m}$

$= 4 \times 10^{-8}\sqrt{m}$

Number of electrons striking per second is

$$n = \frac{q}{e} = \frac{It}{e} = \frac{50 \times 10^{-6} \times 1}{1.6 \times 10^{-19}}$$

$\Rightarrow \quad n = 31.25 \times 10^{13}$

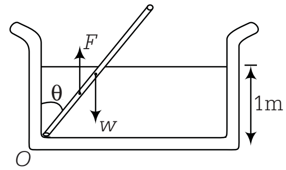
Force = Change of momentum per second

$= n\,(mv) = 31.25 \times 10^{13} \times 4 \times 10^{-8}\sqrt{m}$

$= 125 \times 10^5\sqrt{9.1 \times 10^{-31}} = 1.1924 \times 10^{-8}$ N

47. Length of rod inside the water

$= 1\sec\theta = \sec\theta$



Upthrust, $\quad F = \left(\frac{2}{2}\right)(\sec\theta)\left(\frac{1}{500}\right)(1000)(10)$

$\Rightarrow \quad F = 20\sec\theta$

Weight of rod, $w = 20 \times 10 = 20$ N

For rotational equilibrium of rod, net torque about should be zero.

$\therefore F\left(\frac{\sec\theta}{2}\right)(\sin\theta) = w = (1\sin\theta)$

$\Rightarrow \quad \frac{20}{2}\sec^2\theta = 20$

$\Rightarrow \quad \theta = 45°$

$\Rightarrow \quad F = 20\sec 45°$

$F = 20\sqrt{2}$ N

For vertical equilibrium of rod, force exerted by the hinge on the rod will be $(20\sqrt{2} - 20)$ N downwards i.e. 8.3 N downwards

48. At the two points of the trajectory during projectile motion, the horizontal component of velocity is same. Then,

$$150 \times \frac{1}{2} = v \times \frac{1}{\sqrt{2}} \quad \Rightarrow \quad v = \frac{150}{\sqrt{2}}\ \text{ms}^{-1}$$

Initially, $\quad u_y = u\sin 60 = \frac{150\sqrt{3}}{2}\ \text{ms}^{-1}$

Finally, $\quad v_y = v\sin 45° = \frac{150}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = \frac{150}{2}\ \text{ms}^{-1}$

But $\quad v_y = u_y + a_y t$

$$\frac{150}{2} = \frac{150\sqrt{3}}{2} - 10t$$

$$\Rightarrow \quad 10t = \frac{150\,(\sqrt{3} - 1)}{2}$$

$$\Rightarrow \quad t = 7.5\,(\sqrt{3} - 1)\ \text{s}$$

49. Let R be the radius of sphere, V its volume and ρ its density. Then,

$$\Delta R = R\,\alpha\,\Delta\theta$$

and percentage change in radius

$$\frac{\Delta R}{R} \times 100 = 100\,\alpha\,\Delta\theta \qquad \ldots\text{(i)}$$

Now, $\Delta V = \gamma v \Delta\theta = 3\alpha v \Delta\theta$ $\qquad \{\because \gamma = 3\alpha\}$

$\therefore$ Percentage increase in volume

$$= \frac{\Delta V}{V} \times 100 = 300\ \alpha\Delta\theta \qquad \ldots(ii)$$

Again, $\rho' = \dfrac{\rho}{1 + \gamma\Delta\theta} = \dfrac{\rho}{1 + 3\alpha\Delta\theta}$

$$\therefore \qquad \Delta\rho = \rho - \rho'$$

$$= \rho\left[1 - \frac{1}{1 + 3\ \alpha\Delta\theta}\right] = \frac{(3\alpha\Delta\theta)\,\rho}{1 + 3\alpha\Delta\theta}$$

Percentage change in density

$$= \frac{\Delta\rho}{\rho} \times 100 = \frac{300\ \alpha\Delta\theta}{1 + 3\alpha\Delta\theta} \qquad \ldots(iii)$$

From Eqs. (i), (ii) and (iii), we see that the percentage change is maximum in volume.

50. Since, we know that in Young's double slit experiment.

$$\beta = \frac{D\lambda}{d} \qquad \ldots(i)$$

$$\beta' = \frac{D'\lambda}{d} \qquad \ldots(ii)$$

Subtract Eq. (ii) from Eq. (i), we get

$$\beta - \beta' = \frac{\lambda\,(D - D')}{d}$$

Substituting the given values, we get

$$3 \times 10^{-5} = \frac{\lambda \times 5 \times 10^{-2}}{1 \times 10^{-3}}$$

$$\Rightarrow \qquad \lambda = \frac{3 \times 10^{-8}}{5 \times 10^{-2}} = 0.6 \times 10^{-6}\,\text{m}$$

$$\Rightarrow \qquad \lambda = 600 \times 10^{-9}\,\text{m}$$

$$\Rightarrow \qquad \lambda = 600\ \text{nm}$$

Chemistry

1. Number of moles $= \dfrac{\text{Mass}}{\text{Molecular mass}}$

Given, mass of $Al_2(SO_4)_3 = 50$ g

Molecular mass of

$Al_2(SO_4)_3 = 2 \times 27 + 3(32 + 4 \times 16) = 342$ g

$\therefore$ moles of $Al_2(SO_4)_3 = \dfrac{50}{342} = 0.14$ mol

2. Ionisation energy of $He^+ = 13.6 \times Z^2 eV$

$$= 13.6 \times (2)^2 eV$$

$$= 13.6 \times 4\ eV = 54.4\ eV$$

3. (a) Metallic radii increase in a group from top to bottom. Thus, $Li < Na < K < Rb$ is true.

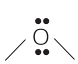
(b) Electron gain enthalpy of $Cl > F$ and afterwards it decreases along a group.

(c) Ionisation enthalpy increase along a period left to right but due to presence of half-filled orbital in N, ionisation enthalpy of $N > O$. Thus $B < C < N < O$ is incorrect.

(d) For isoelectronic species,

ionic size $\propto \dfrac{1}{\text{atomic number}}$

Thus, $Al^{3+} < Mg^{2+} < Na^+ < F^-$ is true.

4. The H  H bond angle in H_2O is 104.5° due to presence of two lone pair of electrons. This fact can be explained with the help of valence shell electron pair repulsion theory (VSEPR).

5. Van der Waals' gas approaches ideal behaviour at extremely low pressure and high temperature.

6. I : $C(s) + O_2(g) \longrightarrow CO_2$; $\Delta H = -393.5$ kJ/mol

II : $CO(g) + \dfrac{1}{2}O_2(g) \longrightarrow CO_2(g)$;

$$\Delta H = -283.0\ \text{kJ/mol}$$

On substracting II from I

III : $C(s) + \dfrac{1}{2}O_2(g) \longrightarrow CO(g)$;

$$\Delta H = -110.5\ \text{kJ/mol}$$

This equation III also represents formation of 1 mole of CO and thus enthalpy of formation of CO (g) is – 110.5 kJ / mol

7.

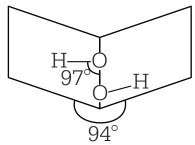
	H_2	+ I_2	$\rightleftharpoons$ 2HI
At initial	1	1	0
At equili	1 – 0.8 = 0.2	1 – 0.8 = 0.2	2 × 0.8 = 1.6

$$\because \qquad K_C = \frac{[HI]^2}{[H_2]\,[I_2]}$$

$$\therefore \qquad K_C = \frac{1.6 \times 1.6}{0.2 \times 0.2}$$

$$\Rightarrow \qquad K_C = 64$$

8. H_2O_2 is a pale blue liquid, it can be oxidised by ozone. H_2O_2 acts as both oxidising and reducing agent. The value of dipole moment of H_2O_2 is 2.1 D. Which suggest it cannot be planar infact it has open book like structure.



Structure of H_2O_2

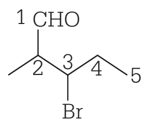
Two O — H lie in different plane

9. KO_2 absorb CO_2 and increase O_2 concentration. So, it is used in space and submarines.

10. $BI_3 > BBr_3 > BCl_3 > BF_3$

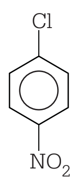
This order can be easily explained on the basis of the tendaney of the halogen atom to back donate its lone pair of electrones to empty p–orbital of the boron atom through $p\pi$-$p\pi$ bonding.

11.



3-bromo-2-methylpentanal

12. Nucleophile always attacks on electron deficient site. Presence of electron withdrawing group such as NO_2 , CHO etc., decreases the electron density on benzene nucleous, hence such group activates the ring towards nucleophilic attack. While presence of electron releasing groups such as *R* or O*R* increase the electron density, thus deactivate the nucleus toward nucleophilie attack. NO_2 group activate the ring more than Cl toward nucleophilic attack hence react readily with nucleophile.



13. Ethyl acetoacetate shows tautomerism.

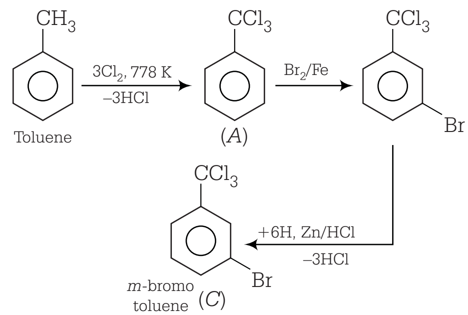
$$CH_3-\overset{\overset{\displaystyle O}{\|}}{C}-CH_2-COOC_2H_5$$

Keto form

$$\longleftrightarrow CH_3-\overset{\overset{\displaystyle OH}{|}}{C}=CH-COOC_2H_5.$$

Enol form

14.



15. We know that,

$$O_3 + CCl_2F_2 \longrightarrow 2OCl + OF_2$$

Thus, in this reaction OCl is produced.

16. Sodium chloride (NaCl) has face centred cubic structure. It contains 4 Na^+ and 4 Cl^- ions. Each Na^+ ino is surrounded by 8 Cl^- ions and each Cl^- ion is also surrounded by 8 Na^+ ions.

17. $$\text{Molarity} = \frac{\text{\% by weight of solute} \times \text{density of solution} \times 10 \text{ (in litre)}}{\text{molecular weight of the solute}}$$

$$\text{Molarity} = \frac{40 \times 1.2 \times 10}{M \times 1000}$$

Again, since

$$\text{Molarity} = \frac{\text{weight of the solute}/M}{\text{volume of the solution (in litre)}}$$

Therefore, weight of solute = 480 g

18. $$E_{cell} = E^\circ_{(cathode)} - E^\circ_{(anode)}$$
$$= 0.41 - 0.76 = -0.35 \text{ V}$$

19. $$N_2(g) + 3H_2(g) \rightleftharpoons 2\,NH_3(g); +22 \text{ kcal}$$

$\because$ The activation energy for the forward reaction = 50 kcal

$\therefore$ The activation energy for the backward reaction = 50 + 72 = + 72 kcal

20. Coagulating power of an electrolyte for arsenious sulphide decreases as $Al^{3+} > Ba^{2+} > Na^+$

21. $${}^{228}_{88}X \xrightarrow{-3\alpha} {}^{216}_{82}Pb \xrightarrow{-\beta} {}^{216}_{83}Bi\ (Y)$$

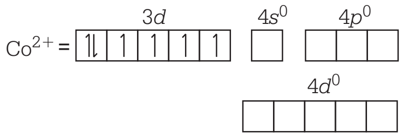
22. Hydrolysis of NCl_3 gives NH_3 or NH_4OH and HClO as

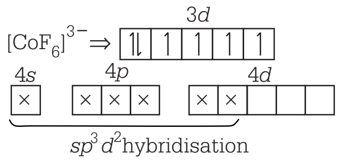
$$NCl_3 + 4H_2O \longrightarrow NH_4OH + 3HOCl$$

23. $4Ag + 8CN^- + 2H_2O + O_2 \longrightarrow 4[Ag(CN)_2]^- + 4OH^-$

This process is called cynide process. It is used in the extraction of silver from argentite (Ag_2S).

24. (a) $[Co\ F_6]^{3-}$ ion

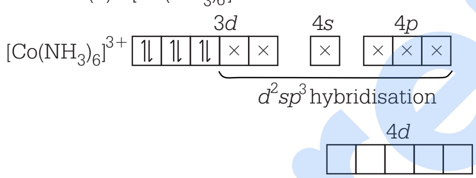




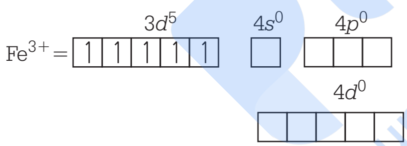
F^- is a weak ligand. It cannot pair up -electrons and forms outer orbital octahedral complex.

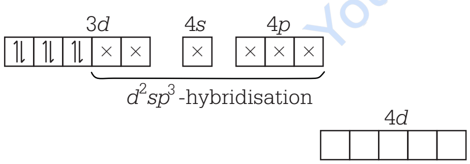
NH_3 and CN^- are strong ligands. So, they pair up -electrons and form their inner orbital complex.

(b) $[Co(NH_3)_6]^{3+}$ ion

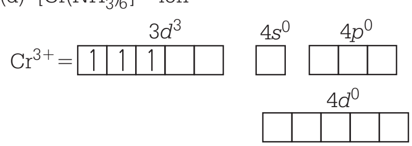


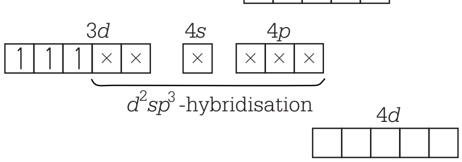
(c) $[Fe(CN)_6]^{3-}$ ion





(d) $[Cr(NH_3)_6]^{3+}$ ion





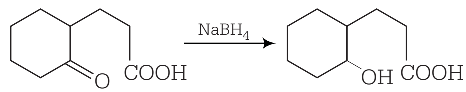
25. $C_2H_5Cl + KCN \xrightarrow{C_2H_5OH} \underset{(X)}{C_2H_5CN} + KCl$

$C_2H_5CN \xrightarrow[\Delta]{H_3O^+,\ 2H_2O} C_2H_5COOH + NH_3$

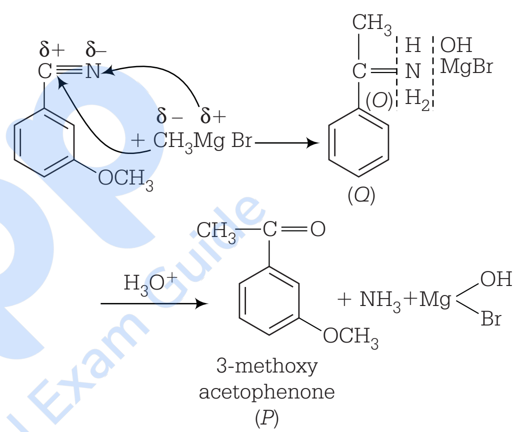
or

$(C_3H_6O_2)$
(y)

26. $NaBH_4$ reduces aldehyde into 1° alcohol,



27.



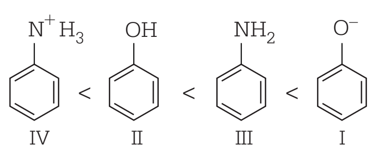
28. (i) Organic compound give an oxime with hydroxyl amine, therefore, it must be an aldehyde or ketone.

(ii) Organic compound did not give silver mirror with Tollen's reagent, therefore it cannot be an aldehyde. Therefore, compound is ketone and its molecular formula with be CH_3COCH_3.

29. $CH_3CH_2I \xrightarrow{NaCN} \underset{(A)}{CH_3CH_2CN} \xrightarrow{OH^-}$

$\underset{(B)}{CH_3CH_2CONH_2} \xrightarrow{Br_2/NaOH} \underset{(C)}{CH_3CH_2NH_2}$

30. Coupling of diazonium salts takes place in the following order as



31. Aldehyde and α-hydroxy ketones give positive Tollen's test. Glucose has an aldehyde group and fructose is an α-hydroxy ketone.

32. Chain growth polymers are formed by the chain growth polymerisation or chain polymerisation. This polymerisation process involves in a series of reaction each of which consumes a relative particles and produces another similar particle resulting a chain reaction. Teflon is a chain growth polymer. It is the polymer of tetraflouro ethylene.

$$\underset{\text{Tetrofluoro ethylene}}{n(CF_2 = CF_2)} \xrightarrow[\text{Pressure}]{\text{High}} \underset{\text{Teflon}}{-\!\!(CF_2-CF_2)_n\!\!-}$$

33. Sodium alkyl benzene sulphonate is used as detergent.

34. Ring test for nitrate ions is

$$FeSO_4 + NO + 5H_2O \longrightarrow \underset{\text{Dark brown ring}}{[Fe(H_2O)_5NO]\,SO_4}$$

35. Zinc is purified by zone-refining method.

36. Perchloric acid ($HClO_4$) is the strongest acid among these acidic because the acidic character of oxoacid increases with increasing the oxidation number of a particular halogen atom.

$$\overset{+1}{HOCl} < \overset{+2}{HClO_2} < \overset{+5}{HClO_3} < \overset{+7}{HClO_4}$$

37. Aerosol is colloidal system of solid in gas, e.g, smoke. So, dispersion medium in aerosol is gas.

38. Given, $k_1 = 10^{10}\, e^{-20,000/T}$

$$k_2 = 10^{12}\, e^{-24,606/T}$$

$$k_1 = k_2 \Rightarrow \; 10^{10}\, e^{-20,000/T} = 10^{12}\, e^{-24,606/T}$$

$$e^{-\frac{20,000}{T}+\frac{24606}{T}} = 10^2$$

$$e^{\frac{4,606}{T}} = 10^2$$

On taking log both sides,

$$\frac{4.606}{2.303\ T} = \log 10^2$$

$$2\log 10 \times T = \frac{4606}{2.303}$$

$$T = \frac{4606}{2.303 \times 2} = \frac{4606}{4.606} = 1000\ K$$

39. $w = 60g,\ i = 5A$

$$\text{Equivalent of Ca} = \frac{\text{atomic weight}}{\text{Valency}} = \frac{40}{2} = 20$$

According to first law of faraday electrolysis

$$w = Zit = \frac{\text{Equivalent weight}}{96500} \times i \times T$$

$$60 = \frac{20}{96500} \times 5 \times T$$

$$T = \frac{96500 \times 60}{20 \times 5}\ s$$

$$T = \frac{96500 \times 60}{20 \times 5 \times 60 \times 60}\ h$$

$$= 16.08\ h. \approx 16\ h.$$

40. Given,

$$T_f = 273 + 16.6 = 289.6\ K$$
$$L_f = 180.75\ J/g$$
$$R = 8.314\ J/K/mol$$
$$K_f = ?,$$

$$\therefore \quad K_f = \frac{RT_f^2}{1000 \times L_f} = \frac{8.314 \times (289.6)^2}{1000 \times 180.75}$$

$$\therefore \quad K_f = 3.86$$

41. The coordination number is 8:8 in $Cs^+ : Cl^-$.

The coordination number is 6:6 in $Na^+ : Cl^-$.

42. $N_2O_4\,(g) \rightleftharpoons 2NO_2\,(g)$

Molar concentration of

$$[N_2O_4] = \frac{9.2}{92} = 0.1\ mol/L$$

In equilibrium state,
when it 50% dissociates

$$[N_2O_4] = 0.05\ M$$
$$[NO_2] = 0.1\ M$$

Now, from $\quad K_C = \dfrac{[NO_2]^2}{[N_2O_4]}$

$$K_C = \frac{0.1 \times 0.1}{0.05} = 0.2$$

43. The Gibbs-Helmholtz equation is as

$$G = H + T\left[\frac{\delta G}{\delta T}\right]_p$$

Dividing above equation by T^2

$$\frac{G}{T^2} = \frac{H}{T^2} + \frac{1}{T}\left[\frac{\delta G}{\delta T}\right]_p$$

This on rearrangement becomes.

$$\left[\frac{\delta(G/T)}{\delta T}\right] = -\frac{H}{T^2}$$

Thus, enthalpy, $H = -T^2\left[\dfrac{\delta G/T}{\delta T}\right]_p$

Where, $\quad H = \text{enthalpy}$

44. $Cl_2O = 2 \times 17 + 16 = 42$ electrons

$ICl_2^- = 53 + 2 \times 17 + 1 = 88$ electrons

$Cl_2^- = 2 \times 17 + 1 = 35$ electrons

$IF_2^+ = 53 + 2 \times 9 - 1 = 70$ electrons
$ClO_2 = 17 + 2 \times 8 = 33$ electrons
$ClO_2 = 17 + 2 \times 8 + 1 = 33$ electrons
$ClO_2^- = 17 + 2 \times 8 + 1 = 34$ electrons
$ClF_2^+ = 17 + 2 \times 9 - 1 = 34$ electrons

ClO_2^- and ClF_2^+ contain 34 electrons each.
Hence they are isoelectronic.

45. Atomic and ionic radii increase from top to bottom in a group due to the inclusion of another shell at every step. Hence, Cs^+ ion will be the largest among given IA group ions, i.e.

$$Na^+, Li^+ \text{ and } K^+.$$

46. In the ground state of an atom, the number of states is limited by Hund's rule. These are $\frac{Ln}{Lr.(n-r)!}$ ways in which electron in orbital may be arranged which do not violate Paulis exclusion principle.

Where, $n =$ number of maximum electrons that can be filled in an orbital

and $r =$ number of electrons present in orbital.

But the valid ground state term is calculated by Hund's rule of maximum multiplicity. As Hund's rule gives the most stable electronic configuration of electrons.

47. $$\underset{2\times 276\text{ g}}{2\,Ag_2CO_3} \xrightarrow[\Delta]{} \underset{4\times 108\text{g}}{4Ag} + 2CO_2 + O_2$$

$\therefore$ 2×276 g of Ag_2CO_3 gives $= 4 \times 108$ g Ag

$\therefore$ 1 g of Ag_2CO_3 will give $= \frac{4 \times 108}{2 \times 276}$

$\therefore$ 2.76 g of Ag_2CO_3 will give $= \frac{4 \times 108 \times 2.76}{2 \times 276}$

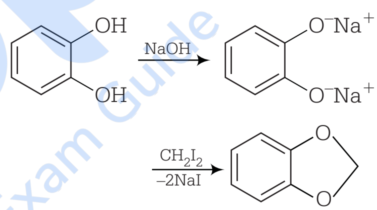
$= 2.16$ g

48. $$\underset{\displaystyle |\atop CH_2Br}{CH_2Br} + 2\underset{\text{Alc.}}{KOH} \xrightarrow{\Delta} \underset{\text{Acetylene}}{\underset{\displaystyle |||\atop CH}{CH}} + 2KBr + 2H_2O$$

This is dehydrohalogenation reaction.

49. I_2 form complex ion I_3^- in KI solution due to which it dissolve in it.

50. Catechol is most acidic out of all dihydric phenols.



The reaction is Williamson's synthesis type reaction.

Mathematics

1. Let $y = x^{3000}$, $x = 1$ and $x + \Delta x = 1.0002$

Then, $\Delta x = 1.0002 - 1 = 0.0002$

Also, $y = 1$, when $x = 1$

Now, $y = x^{3000}$

$\Rightarrow \frac{dy}{dx} = 3000\, x^{2999}$

$\Rightarrow \left[\frac{dy}{dx}\right]_{x=1} = 3000$

$\because \Delta y = \frac{dy}{dx} \Delta x$

$\therefore \Delta y = 3000 \times 0.0002 = \frac{6}{10} = 0.6$

Hence, $(1.0002)^{3000} = y + \Delta y = 1 + 0.6 = 1.6$

2. Let $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$.

Then, $\mathbf{a} \cdot \hat{\mathbf{i}} = a_1, \mathbf{a} \cdot \hat{\mathbf{j}} = a_2, \mathbf{a} \cdot \hat{\mathbf{k}} = a_3$

$\mathbf{a} \times \hat{\mathbf{i}} = -a_2\hat{\mathbf{k}} + a_3\hat{\mathbf{j}}, \mathbf{a} \times \hat{\mathbf{j}} = a_1\hat{\mathbf{k}} - a_3\hat{\mathbf{i}}$

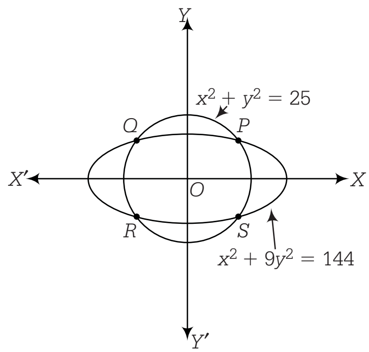
and $\mathbf{a} \times \hat{\mathbf{k}} = -a_1\hat{\mathbf{j}} + a_2\hat{\mathbf{i}}$

$\therefore (\mathbf{a} \cdot \hat{\mathbf{i}})(\mathbf{a} \times \hat{\mathbf{i}}) + (\mathbf{a} \cdot \hat{\mathbf{j}})(\mathbf{a} \times \hat{\mathbf{j}}) + (\mathbf{a} \cdot \hat{\mathbf{k}})(\mathbf{a} \times \hat{\mathbf{k}})$

$= a_1(-a_2\hat{\mathbf{k}} + a_3\hat{\mathbf{j}}) + a_2(a_1\hat{\mathbf{k}} - a_3\hat{\mathbf{i}}) + a_3(-a_1\hat{\mathbf{j}} + a_2\hat{\mathbf{i}})$

$= \mathbf{0}$

3. Clearly, A is the set of all points on the circle $x^2 + y^2 = 25$ and B is the set of all points on the ellipse $x^2 + 9y^2 = 144$. These two intersect at four points P, Q, R and S.



Hence, $A \cap B$ contains four points.

4. Clearly, $f(x) = \sqrt{x^2 - 4}$ is continuous on [2, 3] and differentiable on (2, 3).

So, by Lagrange's mean value theorem, there exists $c \in (2, 3)$ such that

$$f'(c) = \frac{f(3) - f(2)}{3 - 2}$$

$$\Rightarrow \quad \frac{c}{\sqrt{c^2 - 4}} = \sqrt{5} - 0$$

$$\left[\because f(x) = \sqrt{x^2 - 4} \Rightarrow f'(x) = \frac{x}{\sqrt{x^2 - 4}}\right]$$

$$\Rightarrow \quad c^2 = 5(c^2 - 4)$$

$$\Rightarrow \quad 4c^2 = 20$$

$$\Rightarrow \quad c = \sqrt{5}$$

5. The mean of the series $a, a + d, a + 2d, \ldots a + 2nd$ is $\bar{x} = \frac{1}{2n + 1}[a + a + d + a + 2d + \ldots + a + 2nd]$

$$\Rightarrow \bar{x} = \frac{1}{2n + 1}\left\{\frac{2n + 1}{2}(a + a + 2nd)\right\} = a + nd$$

$\therefore$ Mean deviation from mean,

$$MD = \frac{1}{2n + 1}\sum_{r=0}^{2n}|(a + rd) - (a + nd)|$$

$$\Rightarrow \quad MD = \frac{1}{2n + 1}\sum_{r=0}^{2n}|r - n|\, d$$

$$\Rightarrow \quad MD = \frac{1}{2n + 1}\{2d(1 + 2 + \ldots + n)\} = \frac{n(n + 1)d}{2n + 1}$$

6. We have,

$$f(x) = xe^{x(1 - x)}$$

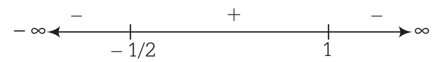
On differentiating both sides w.r.t. , we get

$$f'(x) = e^{x(1 - x)} + x(1 - 2x)e^{x(1 - x)}$$

$$\Rightarrow \quad f'(x) = (1 + x - 2x^2)e^{x(1 - x)}$$

$$\Rightarrow \quad f'(x) = -(x - 1)(2x + 1)e^{x(1 - x)}$$

Since, $e^{x(1 - x)} > 0$ for all x. Therefore, signs of $f'(x)$ for different values of x are as shown below:



Clearly, f(x) is increasing on $\left[-\frac{1}{2}, 1\right]$ and decreasing on $\left(-\infty, -\frac{1}{2}\right] \cup [1, \infty)$.

7. We have,

$$(1 + \omega)(1 + \omega^2)(1 + \omega^3)(1 + \omega^4)\ldots(1 + \omega^{3n})$$

$$= [\{(1 + \omega)(1 + \omega^2)\}\{(1 + \omega^4)(1 + \omega^5)\}\ldots$$
$$\{(1 + \omega^{3n-2})(1 + \omega^{3n-1})\} \times (1 + \omega^3)(1 + \omega^6)$$
$$(1 + \omega^9)\ldots(1 + \omega^{3n})]$$

$$= 2^n[\{(-\omega^2)(-\omega)\}\{(-\omega^2)(-\omega)]\ldots\{(-\omega^2)(-\omega)\}]$$

$$\Rightarrow [(1 + w)(1 + w^2)\{(1 + w)(1 + w^2)\}\{(1 + w)(1 + w^2)\}$$
$$\ldots\{(1 + w)(1 + w^2)\}] \times (1 + 1)(1 + 1)\ldots(1 + 1)$$

$$= 2^n$$

8. Clearly, X is a binomial variate with parametres n and $p = \frac{1}{2}$ such that

$$P(X = r) = {}^nC_r p^r q^{n-r} = {}^nC_r\left(\frac{1}{2}\right)^r\left(\frac{1}{2}\right)^{n-r}$$

$$= {}^nC_r\left(\frac{1}{2}\right)^n$$

Now, $P(X = 4)$, $P(X = 5)$ and $P(X = 6)$ are in AP.

$$\Rightarrow \quad 2P(X = 5) = P(X = 4) + P(X = 6)$$

$$\Rightarrow \quad 2 \cdot {}^nC_5\left(\frac{1}{2}\right)^n = {}^nC_4\left(\frac{1}{2}\right)^n + {}^nC_6\left(\frac{1}{2}\right)^n$$

$$\Rightarrow \quad 2 \cdot {}^nC_5 = {}^nC_4 + {}^nC_6$$

$$\Rightarrow \quad 2 \cdot \frac{n!}{(n - 5)!\,5!} = \frac{n!}{(n - 4)!\,4!} + \frac{n!}{(n - 6)!\,6!}$$

$$\Rightarrow \quad \frac{2}{5(n - 5)} = \frac{1}{(n - 4)(n - 5)} + \frac{1}{6 \times 5}$$

$$\Rightarrow \quad n^2 - 21n + 98 = 0$$

$$\Rightarrow \quad (n - 7)(n - 14) = 0$$

$$\therefore \quad n = 7 \text{ or } 14$$

9. Suppose $(s + 1)$th term contains x^{2r}.

Then, we have

$$T_{s+1} = {}^{n-3}C_s x^{n-3-s}\left(\frac{1}{x^2}\right)^s$$

$$= {}^{n-3}C_s x^{n-3-3s}$$

This will contain x^{2r}, if

$$n-3-3s=2r$$

$$\Rightarrow \quad s=\frac{n-3-2r}{3}$$

$$\Rightarrow \quad s=\frac{n-2r}{3}-1$$

$$\Rightarrow \quad s+1=\frac{n-2r}{3}$$

$$\Rightarrow \quad n-2r=3(s+1)$$

Hence, $(n-2r)$ is a positive integral multiple of 3.

10. We have, $\sin^{-1}\left\{\cot\left(\sin^{-1}\sqrt{\frac{2-\sqrt{3}}{4}}+\cos^{-1}\frac{\sqrt{12}}{4}+\sec^{-1}\sqrt{2}\right)\right\}$

$$=\sin^{-1}\left\{\cot\left(\sin^{-1}\frac{\sqrt{3}-1}{2\sqrt{2}}+\cos^{-1}\frac{\sqrt{3}}{2}+\sec^{-1}\sqrt{2}\right)\right\}$$

$$=\sin^{-1}\left\{\cot\left(\frac{\pi}{12}+\frac{\pi}{6}+\frac{\pi}{4}\right)\right\}$$

$$=\sin^{-1}\left(\cot\frac{\pi}{2}\right)$$

$$=\sin^{-1}0=0$$

11. We have,

$$\cos^{-1}\frac{x}{2}+\cos^{-1}\frac{y}{3}=\theta$$

$$\Rightarrow \quad \cos^{-1}\left\{\frac{xy}{6}-\sqrt{1-\frac{x^2}{4}}\cdot\sqrt{1-\frac{y^2}{9}}\right\}=\theta$$

$$\Rightarrow \quad xy-\sqrt{4-x^2}\cdot\sqrt{9-y^2}=6\cos\theta$$

$$\Rightarrow \quad (xy-6\cos\theta)^2=(4-x^2)(9-y^2)$$

$$\Rightarrow \quad -12xy\cos\theta+36\cos^2\theta=36-4y^2-9x^2$$

$$\Rightarrow \quad 9x^2+4y^2-12xy\cos\theta=36\sin^2\theta$$

12. We have,

$$\tan(\sec^{-1}x)=\sin\left(\cos^{-1}\frac{1}{\sqrt{5}}\right)$$

$$\Rightarrow \quad \sqrt{\sec^2(\sec^{-1}x)-1}=\sqrt{1-\cos^2\left(\cos^{-1}\frac{1}{\sqrt{5}}\right)}$$

$$\Rightarrow \quad \sqrt{x^2-1}=\sqrt{1-\frac{1}{5}}$$

$$\Rightarrow \quad x^2-1=\frac{4}{5}$$

$$\Rightarrow \quad x^2=\frac{9}{5}$$

$$\Rightarrow \quad x=\pm\frac{3}{\sqrt{5}}$$

13. We have,

$$y=(2x-1)e^{2(1-x)}$$

On differentiating both sides w.r.t. , we get

$$\frac{dy}{dx}=2e^{2(1-x)}-2(2x-1)e^{2(1-x)}$$

$$\Rightarrow \quad \frac{dy}{dx}=2e^{2(1-x)}(2-2x)=4e^{2(1-x)}(1-x)$$

At points of maximum, we must have

$$\frac{dy}{dx}=0\Rightarrow x=1$$

Now, $$\frac{d^2y}{dx^2}=-8e^{2(1-x)}(1-x)-4e^{2(1-x)}$$

$$\Rightarrow \quad \left[\frac{d^2y}{dx^2}\right]_{x=1}=-4<0$$

So, y is maximum at $x=1$. Clearly, $y=1$ for $x=1$.

Thus, the point of maximum is $(1,1)$.

The equation of the tangent at $(1,1)$ is

$$y-1=0\,(x-1)$$

$$\Rightarrow \quad y=1$$

14. The truth table of $(p\to\sim p)\wedge(\sim p\to p)$ is as follows:

p	$\sim p$	$p\to\sim p$	$\sim p\to p$	$(P\to\sim P)\wedge(\sim p\to p)$
T	F	F	T	F
F	T	T	F	F

Clearly, last column of the above truth table contains F only. So, the given statement is a contradiction.

15. There are 6 letters in the word 'DEGREE', namely 3E's and D, G, R. Four letters out of six can be selected in the following ways:

(i) 3 like letters and 1 different, viz. EEE + D, G, R

(ii) 2 like letters and 2 different, viz. EEE + any two of D, G, R

(iii) All different letters, viz. EDGR

So, the total number of ways

$$={}^3C_1+{}^3C_2+1=3+3+1=7$$

16. Total number of triangles = Number of triangles with vertices on sides (PQ, QR, RS + QR, RS, PS + RS, PS, PQ + PS, PQ, QR)

$$= {}^3C_1 \times {}^4C_1 \times {}^5C_1 + {}^4C_1 \times {}^5C_1 \times {}^6C_1 + {}^5C_1 \times {}^6C_1 \times {}^3C_1 + {}^6C_1 \times {}^3C_1 \times {}^4C_1$$

$$= 60 + 120 + 90 + 72 = 342$$

17. We observe that

$$\frac{x^4 + x^2 + x + 1}{x^2 - x + 1} \to \frac{2}{3} \text{ as } x \to -1$$

But $\dfrac{1 - \cos(x+1)}{(x+1)^2} \to \dfrac{0}{0}$ as $x \to -1$

$$\therefore \lim_{x \to -1} \left(\frac{x^4 + x^2 + x + 1}{x^2 - x + 1}\right)^{\frac{1-\cos(x+1)}{(x+1)^2}}$$

$$= \left(\frac{2}{3}\right)^{\lim_{x\to -1} \frac{1-\cos(x+1)}{(x+1)^2}} = \left(\frac{2}{3}\right)^{\lim_{x\to -1} \frac{2\sin^2\left(\frac{x+1}{2}\right)}{4\left(\frac{x+1}{2}\right)^2}} = \left(\frac{2}{3}\right)^{1/2}$$

18. We have,

$$\left(x - \frac{1}{x}\right)^4 \left(x + \frac{1}{x}\right)^3$$

$$= \left({}^4C_0\, x^4 - {}^4C_1 x^2 + {}^4C_2 - {}^4C_3 \frac{1}{x^2} + {}^4C_4 \frac{1}{x^4}\right)$$

$$\times \left({}^3C_0 x^3 + {}^3C_1 x + {}^3C_2 \frac{1}{x} + {}^3C_3 \frac{1}{x^3}\right)$$

Clearly, there is no term free from x on RHS. Therefore, the term independent of x on LHS is zero.

19. We have,

$$A^2 = A$$

$\therefore$ $(I + A)^2 = (I + A)(I + A) = I + 2A + A^2 = I + 3A$

and $(I + A)^3 = (I + A)^2 (I + A)$

$= (I + 3A)(I + A)$ $[\because (I + A)^2 = I + 3A]$

$= I + 4A + 3A^2$

$= I + 7A$ $[\because A^2 = A]$

Thus, we have

$$(I + A)^2 = I + 3A \text{ and } (I + A)^3 = I + 7A$$

$\Rightarrow$ $(I + A)^2 = I + (2^2 - 1)A$

and $(I + A)^3 = I + (2^3 - 1)A$

Hence, $(I + A)^n = I + (2^n - 1)A$

$\therefore$ $(I + A)^n = I + \lambda A$

$\Rightarrow$ $\lambda = 2^n - 1$

20. We have, $u = e^x \sin x$ and $v = e^x \cos x$

On differentiating both sides w.r.t. , we get

$$\frac{du}{dx} = e^x \sin x + e^x \cos x = u + v$$

and $\dfrac{dv}{dx} = e^x \cos x - e^x \sin x = v - u$

$$\therefore v\frac{du}{dx} - u\frac{dv}{dx} = v(u + v) - u(v - u) = u^2 + v^2$$

Now, $\dfrac{d^2u}{dx^2} = \dfrac{du}{dx} + \dfrac{dv}{dx} = u + v + v - u = 2v$

and $\dfrac{d^2v}{dx^2} = \dfrac{dv}{dx} - \dfrac{du}{dx}$

$= (v - u) - (v + u) = -2u$

21. Let $y = \tan^{-1}\left(\dfrac{\sqrt{1 + x^2} - 1}{x}\right)$

Putting $x = \tan\theta$, we get

$$y = \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right) = \tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$= \frac{1}{2}\tan^{-1} x$$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{2(1 + x^2)}$$

and $z = \tan^{-1}\left(\dfrac{2x\sqrt{1 - x^2}}{1 - 2x^2}\right)$

Putting $x = \sin\theta$, we get

$$z = \tan^{-1}\left(\frac{2\sin\theta\cos\theta}{\cos 2\theta}\right) = \tan^{-1}(\tan 2\theta)$$

$\Rightarrow$ $z = 2\theta = 2\sin^{-1} x$

On differentiating both sides w.r.t. x, we get

$$\frac{dz}{dx} = \frac{2}{\sqrt{1 - x^2}}$$

Thus, $\dfrac{dy}{dz} = \dfrac{dy/dx}{dz/dx} = \dfrac{1}{4(1 + x^2)}\sqrt{1 - x^2}$

$$\therefore \left[\frac{dy}{dz}\right]_{x=0} = \frac{1}{4}$$

22. Let $I = \displaystyle\int \frac{1 - x^2}{(1 + x^2)\sqrt{1 + x^4}}\, dx$

$$= \int \frac{\frac{1}{x^2} - 1}{\left(x + \frac{1}{x}\right)\sqrt{x^2 + \frac{1}{x^2}}}\, dx$$

$$= -\int \frac{1}{\left(x+\frac{1}{x}\right)\sqrt{\left(x+\frac{1}{x}\right)^2-(\sqrt{2})^2}}\, d\left(x+\frac{1}{x}\right)$$

$$= \frac{1}{\sqrt{2}} \operatorname{cosec}^{-1}\left(\frac{x+\frac{1}{x}}{\sqrt{2}}\right) + C$$

$$= \frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{\sqrt{2}x}{x^2+1}\right) + C$$

23. We have,

$$\begin{vmatrix} 1 & a & b \\ 1 & c & a \\ 1 & b & c \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1 & a & b \\ 0 & c-a & a-b \\ 0 & b-a & c-b \end{vmatrix} = 0$$

[applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$]

$\Rightarrow \quad (c-a)(c-b)+(a-b)^2 = 0$

$\Rightarrow \quad a^2+b^2+c^2-ab-bc-ca = 0$

$\Rightarrow 2a^2+2b^2+2c^2-2ab-2bc-2ca = 0$

$\Rightarrow \quad (a-b)^2+(b-c)^2+(c-a)^2 = 0$

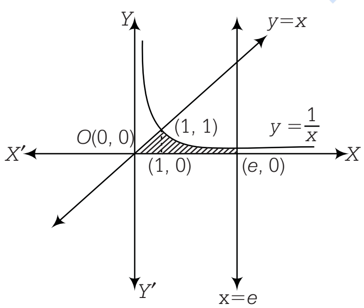
$\Rightarrow \quad a = b = c$

$\therefore$ ΔABC is an equilateral triangle.

$\Rightarrow \quad A = B = C = \dfrac{\pi}{3}$

$\therefore \quad \sin^2 A + \sin^2 B + \sin^2 C = 3\sin^2\dfrac{\pi}{3} = \dfrac{9}{4}$

24. Required area $= \int_0^1 x\,dx + \int_1^e \frac{1}{x}dx$



$$= \left[\frac{x^2}{2}\right]_0^1 + [\log x]_1^e = \frac{1}{2}+1 = \frac{3}{2}$$

25. Let $x = \dfrac{1}{y}$. Then,

$$\lim_{x\to\infty}\left\{\frac{a_1^{1/x}+a_2^{1/x}+\ldots+a_n^{1/x}}{n}\right\}^{nx}$$

$$= \lim_{y\to 0}\left\{\frac{a_1^y+a_2^y+\ldots+a_n^y}{n}\right\}^{n/y}$$

$$= \lim_{y\to 0}\left\{1+\frac{a_1^y+a_2^y+\ldots+a_n^y-n}{n}\right\}^{n/y}$$

$$= e^{\lim_{y\to 0}\left\{\frac{a_1^y-1}{y}+\frac{a_2^y-1}{y}+\ldots+\frac{a_n^y-1}{y}\right\}}$$

$$= e^{\log a_1 + \log a_2 + \ldots + \log a_n}$$

$$= e^{\log(a_1 a_2 \ldots a_n)}$$

$$= a_1 a_2 a_3 \ldots a_n$$

26. Let $I = \displaystyle\int \frac{\sec^2 x}{(\sec x+\tan x)^{9/2}}dx$

Put $\sec x + \tan x = t$

Then, $\quad \sec x - \tan x = \dfrac{1}{t}$

and $\quad \sec x(\sec x+\tan x)\,dx = dt$

$\Rightarrow \quad \sec x\,dx = \dfrac{1}{t}dt$

Also, $\quad \sec x = \dfrac{1}{2}\left(t+\dfrac{1}{t}\right)$

$$\therefore \quad I = \frac{1}{2}\int \frac{\frac{1}{t}\left(t+\frac{1}{t}\right)}{t^{9/2}}dt$$

$$\Rightarrow \quad I = \frac{1}{2}\int \frac{1}{t^{9/2}}+\frac{1}{t^{13/2}}\,dt$$

$$\Rightarrow \quad I = -\frac{1}{7t^{7/2}}-\frac{1}{11t^{11/2}}+K$$

$$\Rightarrow \quad I = -\frac{1}{t^{11/2}}\left\{\frac{t^2}{7}+\frac{1}{11}\right\}+K$$

$$\Rightarrow \quad I = -\frac{1}{(\sec x+\tan x)^{11/2}}\left\{\frac{1}{11}+\frac{1}{7}(\sec x+\tan x)^2\right\}+K$$

27. Let the equation of the circle be

$$x^2+y^2+2gx+2fy+c = 0$$

which passes through (0, 0) and (1, 0).

$\therefore \quad c = 0$ and $1+2g+c = 0$

$\Rightarrow \quad c = 0,\ g = -\dfrac{1}{2}$

The circle $x^2 + y^2 + 2gx + 2fy + c = 0$ touches the circle $x^2 + y^2 = 9$.

$$\therefore \quad \sqrt{g^2 + f^2} = 3 \pm \sqrt{g^2 + f^2 - c}$$

$$[\because C_1C_2 = r_1 \pm r_2]$$

$$\Rightarrow \quad \sqrt{\frac{1}{4} + f^2} = 3 \pm \sqrt{\frac{1}{4} + f^2}$$

$$\Rightarrow \quad 3 = 2\sqrt{\frac{1}{4} + f^2}$$

$$\Rightarrow \quad f^2 = \frac{9}{4} - \frac{1}{4} = 2$$

$$\Rightarrow \quad f = \pm\sqrt{2}$$

Hence, the coordinates of the centre are $\left(\frac{1}{2}, \pm\sqrt{2}\right)$.

28. Given equation of parabola is

$$y^2 - kx + 8 = 0$$

$$\Rightarrow \quad y^2 = k\left(x - \frac{8}{k}\right)$$

$$\Rightarrow \quad (y - 0)^2 = k\left(x - \frac{8}{k}\right)$$

The equation of the directrix of this parabola is

$$x - \frac{8}{k} = -\frac{k}{4} \qquad [\because x = -a]$$

$$\Rightarrow \quad x = \frac{8}{k} - \frac{k}{4}$$

But the equation of the directrix is given as $x - 1 = 0$.

$$\therefore \quad \frac{8}{k} - \frac{k}{4} = 1$$

$$\Rightarrow \quad k^2 + 4k - 32 = 0$$

$$\Rightarrow \quad (k - 4)(k + 8) = 0$$

$$\therefore \quad k = -8, 4$$

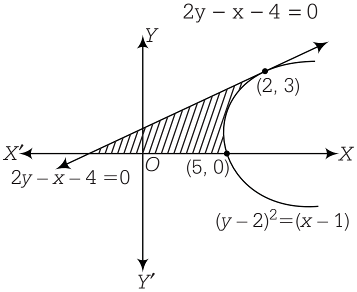
29. Given equation of parabola is

$$y^2 - 4y - x + 5 = 0$$

Then, the equation of tangent at $(2, 3)$ is

$$3y - 2(y + 3) - \frac{(x + 2)}{2} + 5 = 0$$

$$\Rightarrow \quad 2y - x - 4 = 0$$



$\therefore$ Required area is given by

$$A = \int_0^3 (x_2 - x_1)\, dy$$

$$= \int_0^3 [\{(y-2)^2 + 1\} - \{2y - 4\}]\, dy$$

$$= \int_0^3 (y^2 - 6y + 9)\, dy = \int_0^3 (3 - y)^2 dy$$

$$= -\left[\frac{(3 - y)^3}{3}\right]_0^3$$

$$\therefore \quad A = 9$$

30. We have,

$$x = e^{xy\frac{dy}{dx}}$$

$$\Rightarrow \quad \log x = xy\frac{dy}{dx}$$

$$\Rightarrow \quad ydy = \frac{\log x}{x}dx$$

$$\Rightarrow \quad ydy = \log x \, d(\log x)$$

On integrating both sides, we get

$$\frac{y^2}{2} = \frac{(\log_e x)^2}{2} + C$$

$$\Rightarrow \quad y^2 = (\log_e x)^2 + 2C$$

$$\Rightarrow \quad y = \pm\sqrt{(\log_e x)^2 + 2C}$$

31. Number of ways of selecting three numbers one on each card $= 100 \times 100 \times 100 = 100^3$

We know that $(2n + 1)$, $(2n^2 + 2n)$ and $(2n^2 + 2n + 1)$ are Pythagorean triplets.

$\therefore$ For $n = 1, 2, 3, 4, 5, 6$, we get the lengths of three sides of a right angled triangle such that its hypotenuse is less than or equal to 100 cm.

Now, one Pythagorean triplet (e.g. 3, 4, 5; 5, 12, 13; etc.) can be chosen in 3! ways.

$\therefore$ Number of ways of selecting 6 Pythagorean triplets $= 6 \times 3!$

Hence, required probability $= \dfrac{6 \times 3!}{100^3}$

32. It is given that $f(x)$ is continuous at $x = \frac{\pi}{2}$.

$$\therefore \quad k = \lim_{x \to \pi/2} f(x)$$

$$\Rightarrow \quad k = \lim_{x \to \pi/2} \frac{\sin(\cos x) - \cos x}{(\pi - 2x)^3}$$

$$\Rightarrow \quad k = \lim_{x \to \pi/2} \frac{\sin(\cos x) - \cos x}{\cos^3 x} \times \frac{\sin^3\left(\frac{\pi}{2} - x\right)}{8\left(\frac{\pi}{2} - x\right)^3}$$

$$\therefore \quad k = -\frac{1}{6} \times \frac{1}{8} = -\frac{1}{48} \quad \left[\because \lim_{x \to 0} \frac{\sin x - x}{x^3} = -\frac{1}{6}\right]$$

33. We have,

$$f(x) = [x] + \left[x + \frac{1}{2}\right] = \begin{cases} 0, & \text{if } 0 < x < \frac{1}{2} \\ 1, & \text{if } x = \frac{1}{2} \\ 1, & \text{if } \frac{1}{2} < x < 1 \end{cases}$$

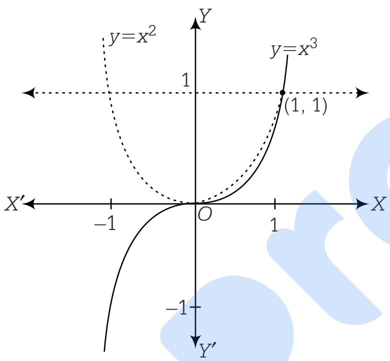
Clearly, $f(x)$ is discontinuous at $x = \frac{1}{2}$.

Also, $\lim_{x \to 1/2^-} f(x) = 0$ and $\lim_{x \to 1/2^+} f(x) = 1$

34. It is evident from the graph of $f(x)$ that

$$f(x) = \begin{cases} 1, & x \geq 1 \\ x^3, & x < 1 \end{cases}$$

Clearly, $f(x)$ is everywhere continuous but it is not differentiable at $x = 1$.



35. We have,

$$x_n = \cos\left(\frac{\pi}{3^n}\right) + i \sin\left(\frac{\pi}{3^n}\right)$$

$\therefore x_1 \cdot x_2 \cdot x_3 \ldots$

$$= \cos\left(\frac{\pi}{3} + \frac{\pi}{3^2} + \ldots\right) + i \sin\left(\frac{\pi}{3} + \frac{\pi}{3^2} + \ldots\right)$$

$$= \cos\left(\frac{\pi/3}{1 - \frac{1}{3}}\right) + i \sin\left(\frac{\pi/3}{1 - \frac{1}{3}}\right)$$

$$= \cos\frac{\pi}{2} + i \sin\frac{\pi}{2} = 0 + i = i$$

36. There can be 2 types of numbers.

(i) Any one of the digits 1, 2, 3, 4 repeats thrice and the remaining digits only once i.e., of the type 1, 2, 3, 4, 4, 4.

(ii) Any two of the digits 1, 2, 3, 4 repeat twice and the remaining two only once i.e., of the type 1, 2, 3, 3, 4, 4.

Number of numbers of the type 1 2 3 4 4 4

$$= \frac{6!}{3!} \times {}^4C_1 = 480$$

Number of numbers of the type 1 2 3 3 4 4

$$= \frac{6!}{2!2!} \times {}^4C_2 = 1080$$

So, the required number $= 480 + 1080 = 1560$

37. We have,

$$9x^2 - 18|x| + 5 = 0$$

$\Rightarrow \quad 9|x|^2 - 18|x| + 5 = 0$

$\Rightarrow \quad (3|x| - 1)(3|x| - 5) = 0$

$\Rightarrow \quad |x| = \frac{1}{3} \text{ or } |x| = \frac{5}{3}$

$\Rightarrow \quad x = \pm\frac{1}{3} \text{ or } x = \pm\frac{5}{3}$

Now, $\log_e\{(x+1)(x+2)\}$ is defined, if

$$(x+1)(x+2) > 0$$

$\Rightarrow \quad x < -2 \text{ or } x > -1$

Clearly, $x = -\frac{5}{3}$ does not satisfy this condition. But all other values of x satisfy the above conditions. Hence, the number of solutions lying in the domain of definition of the given function is 3.

38. We have,

$$2^x + 2^y = 2^{x+y}$$

On differentiating both sides w.r.t. , we get

$$2^x \log 2 + 2^y \log 2 \frac{dy}{dx}$$

$$= 2^{x+y} \log 2 \left(1 + \frac{dy}{dx}\right)$$

$\Rightarrow \quad (2^y - 2^{x+y})\frac{dy}{dx} = 2^{x+y} - 2^x$

$\Rightarrow \quad \frac{dy}{dx} = \frac{2^x(2^y - 1)}{2^y(1 - 2^x)} = 2^{x-y}\left(\frac{2^y - 1}{1 - 2^x}\right)$

39. Since, $3x + y = 0$ is a tangent to the circle with centre at $(2, -1)$.

$\therefore$ Radius = Length of the perpendicular from $(2, -1)$ on $3x + y = 0$

$\Rightarrow \quad \text{Radius} = \frac{6 - 1}{\sqrt{9 + 1}} = \frac{5}{\sqrt{10}} = \sqrt{\frac{5}{2}}$

So, the equation of the circle is

$$(x-2)^2+(y+1)^2=\frac{5}{2}$$

$$\Rightarrow \quad x^2+y^2-4x+2y+\frac{5}{2}=0$$

The combined equation of the tangents drawn from the origin to this circle is

$$SS_1=T^2$$

where, $S=x^2+y^2-4x+2y+\frac{5}{2}$,

$S_1=0^2+0^2-4\times0+2\times0+\frac{5}{2}$

$=\frac{5}{2}$

and $T=x(0)+y(0)-2(x+0)+(y+0)+\frac{5}{2}$

$=-2x+y+\frac{5}{2}$

$$\therefore \left(x^2+y^2-4x+2y+\frac{5}{2}\right)\left(\frac{5}{2}\right)=\left(-2x+y+\frac{5}{2}\right)^2$$

$$\Rightarrow \quad 3x^2-8xy-3y^2=0$$

$$\Rightarrow \quad 3x+y=0 \text{ and } x-3y=0$$

40. We have,

$\sin A+\cos A=m$ and $\sin^3 A+\cos^3 A=n$

Now, $\sin A+\cos A=m$

$\Rightarrow (\sin A+\cos A)^3=m^3$

$\Rightarrow \sin^3 A+\cos^3 A+3\sin A\cos A$

$(\sin A+\cos A)=m^3$

$\Rightarrow n+3\sin A\cos A\ m=m^3 \quad \ldots(i)$

Again, $\sin A+\cos A=m$

$\Rightarrow \sin^2 A+\cos^2 A+2\sin A\cos A=m^2$

$\Rightarrow \sin A\cos A=\frac{m^2-1}{2} \quad \ldots(ii)$

From Eqs. (i) and (ii), we get

$$n+3m\frac{(m^2-1)}{2}=m^3$$

$$\Rightarrow \quad 2n+3m^3-3m=2m^3$$

$$\Rightarrow \quad m^3-3m+2n=0$$

41. We have, $a=4$, $b=3$ and $\angle A=60°$

$$\because \quad \cos A=\frac{b^2+c^2-a^2}{2bc}$$

$$\therefore \quad \cos 60°=\frac{9+c^2-16}{6c}$$

$$\Rightarrow \quad \frac{1}{2}\times6c=c^2-7$$

$$\Rightarrow \quad c^2-3c-7=0$$

42. Let T_r be the th term of the given series. Then,

$$T_r=\frac{2r+1}{1^2+2^2+\ldots+r^2}=\frac{2r+1}{\frac{r(r+1)(2r+1)}{6}}$$

$$=\frac{6}{r(r+1)}$$

$$\Rightarrow \quad T_r=6\left(\frac{1}{r}-\frac{1}{r+1}\right)$$

So, the required sum is given by

$$\sum_{r=1}^{n}T_r=6\sum_{r=1}^{n}\left(\frac{1}{r}-\frac{1}{r+1}\right)$$

$$=6\left[\frac{1}{1}-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\ldots+\frac{1}{n}-\frac{1}{n+1}\right]$$

$$=6\left[1-\frac{1}{n+1}\right]=\frac{6n}{n+1}$$

43. Let $I=\int_{\sqrt{\ln 2}}^{\sqrt{\ln 3}}\frac{x\sin x^2}{\sin x^2+\sin(\ln 6-x^2)}dx$

Putting $x^2=t$

$\Rightarrow 2x\,dx=dt$

$$\therefore \quad I=\frac{1}{2}\int_{\ln 2}^{\ln 3}\frac{\sin t}{\sin t+\sin(\ln 6-t)}dt \quad \ldots(i)$$

Using $\int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx$, we get

$$I=\frac{1}{2}\int_{\ln 2}^{\ln 3}\frac{\sin(\ln 6-t)}{\sin(\ln 6-t)+\sin t}dt \quad \ldots(ii)$$

On adding Eqs.(i) and (ii), we get

$$2I=\int_{\ln 2}^{\ln 3}1\,dt=[t]_{\ln 2}^{\ln 3}$$

$$\Rightarrow \quad 2I=\frac{1}{2}(\ln 3-\ln 2)=\frac{1}{2}\ln\left(\frac{3}{2}\right)$$

$$\Rightarrow \quad I=\frac{1}{4}\ln\left(\frac{3}{2}\right)$$

44. Let $I=\int_1^4 \log_e[x]\,dx$

$$=\int_1^2\log_e 1\,dx+\int_2^3\log_e 2dx+\int_3^4\log_e 3\,dx$$

$$=\log_e 2+\log_e 3=\log_e 6$$

45. We have,

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

The eccentricity e of the ellipse is given by

$$e = \sqrt{1 - \frac{9}{16}} = \frac{\sqrt{7}}{4}$$

So, the coordinates of the foci are $(\pm\sqrt{7}, 0)$.

$\therefore$ Radius of the circle

$$= \sqrt{(\sqrt{7} - 0)^2 + (0 - 3)^2}$$
$$= \sqrt{7 + 9.} = \sqrt{16} = 4$$

46. We have,

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{\sin(e^{x-2} - 1)}{\log(x - 1)}$$

$$= \lim_{x \to 2} \left\{ \frac{\sin(e^{x-2} - 1)}{e^{x-2} - 1} \cdot \frac{e^{x-2} - 1}{x - 2} \cdot \frac{x - 2}{\log[1 + (x - 2)]} \right\}$$

$$= 1 \times 1 \times 1 = 1$$

47. Since, tan A and tan B are the roots of the equation $x^2 - ax + b = 0$. Therefore,

$$\tan A + \tan B = a \text{ and } \tan A \tan B = b$$

Now, $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B} = \dfrac{a}{1 - b}$

Now, $\sin^2(A + B) = \dfrac{1}{2}[1 - \cos 2(A + B)]$

$$\Rightarrow \sin^2(A + B) = \frac{1}{2}\left\{1 - \frac{1 - \tan^2(A + B)}{1 + \tan^2(A + B)}\right\}$$

$$\Rightarrow \sin^2(A + B) = \frac{1}{2}\left\{1 - \frac{1 - \dfrac{a^2}{(1 - b)^2}}{1 + \dfrac{a^2}{(1 - b)^2}}\right\}$$

$$\Rightarrow \sin^2(A + B) = \frac{1}{2}\left\{\frac{2a^2}{a^2 + (1 - b)^2}\right\}$$

$$\Rightarrow \sin^2(A + B) = \frac{a^2}{a^2 + (1 - b)^2}$$

48. We have, $a^2 + b^2 + c^2 = ac + \sqrt{3}\,ab$

$$\Rightarrow \frac{a^2}{4} + c^2 - ac + \frac{3a^2}{4} + b^2 - \sqrt{3}ab = 0$$

$$\Rightarrow \left(\frac{a}{2} - c\right)^2 + \left(\frac{\sqrt{3}a}{2} - b\right)^2 = 0$$

$$\Rightarrow \frac{a}{2} - c = 0 \text{ and } \frac{\sqrt{3}}{2}a - b = 0$$

$$\Rightarrow a = 2c \text{ and } \sqrt{3}a = 2b$$

$$\Rightarrow a = \frac{2b}{\sqrt{3}} = 2c = \lambda \qquad \text{[say]}$$

$$\Rightarrow a = \lambda,\ b = \frac{\sqrt{3}}{2}\lambda \text{ and } c = \frac{\lambda}{2}$$

Now, $b^2 + c^2 = \left(\dfrac{\sqrt{3}\lambda}{2}\right)^2 + \left(\dfrac{\lambda}{2}\right)^2$

$$= \frac{3\lambda^2}{a} + \frac{\lambda^2}{4} = \lambda^2$$

$$\therefore \quad b^2 + c^2 = a^2$$

Hence, the triangle is right angled but not isosceles.

49. We have,

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \frac{1}{3}|\mathbf{b}|\,|\mathbf{c}|\,\mathbf{a}$$

$$\Rightarrow (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\,\mathbf{a} = \frac{1}{3}|\mathbf{b}|\,|\mathbf{c}|\,\mathbf{a}$$

$$\Rightarrow (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - \left\{(\mathbf{b} \cdot \mathbf{c}) + \frac{1}{3}|\mathbf{b}|\,|\mathbf{c}|\right\}\mathbf{a} = \mathbf{0}$$

$$\Rightarrow \mathbf{a} \cdot \mathbf{c} = 0 \text{ and } \mathbf{b} \cdot \mathbf{c} + \frac{1}{3}|\mathbf{b}|\,|\mathbf{c}| = 0$$

[$\because$ **a**, **b** are non-collinear]

$$\Rightarrow |\mathbf{b}|\,|\mathbf{c}|\cos\theta + \frac{1}{3}|\mathbf{b}|\,|\mathbf{c}| = 0$$

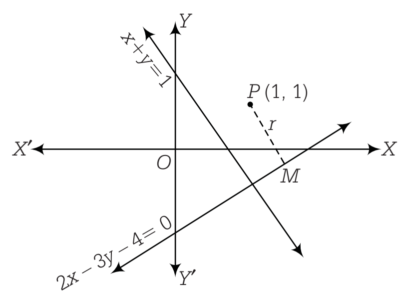
$$\Rightarrow \cos\theta = -\frac{1}{3}$$

$$\Rightarrow \sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - \left(-\frac{1}{3}\right)^2}$$

$$= \sqrt{1 - \frac{1}{9}} = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$$

50. The equation of a line through P (1, 1) and parallel to x + y = 1, is

$$\frac{x - 1}{\cos\frac{3\pi}{4}} = \frac{y - 1}{\sin\frac{3\pi}{4}}$$



Let PM = r.
Then, the coordinates of M given by

$$\frac{x-1}{\cos\frac{3\pi}{4}} = \frac{y-1}{\sin\frac{3\pi}{4}} = r \text{ is } \left(1-\frac{r}{\sqrt{2}}, 1+\frac{r}{\sqrt{2}}\right).$$

$\because$ M lies on $2x - 3y - 4 = 0$.

$$\therefore \quad 2\left(1-\frac{r}{\sqrt{2}}\right) - 3\left(1+\frac{r}{\sqrt{2}}\right) - 4 = 0$$

$$\Rightarrow \quad 2 - \frac{2r}{\sqrt{2}} - 3 - \frac{3r}{\sqrt{2}} - 4 = 0$$

$$\Rightarrow \quad -5 - \frac{5r}{\sqrt{2}} = 0$$

$$\Rightarrow \quad r = -\sqrt{2}$$

Hence, PM = $\sqrt{2}$

51. At the point where the given curve crosses X-axis, we have

$$y = 0$$

$$\Rightarrow \quad ax^2 = 1$$

$$[\text{putting } y = 0 \text{ in } ax^2 + 2hyx + by^2 = 1]$$

$$\Rightarrow \quad x = \pm\frac{1}{\sqrt{a}}$$

Thus, the given curve cuts X-axis at $P\left(\frac{1}{\sqrt{a}}, 0\right)$ and $Q\left(-\frac{1}{\sqrt{a}}, 0\right)$.

Now, $ax^2 + 2hxy + by^2 = 1$

On differentiating both sides w.r.t.   , we get

$$2ax + 2h\left(x\frac{dy}{dx} + y\right) + 2by\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{ax+hy}{hx+by}$$

$$\Rightarrow \quad \left[\frac{dy}{dx}\right]_P = -\frac{a}{h} \text{ and } \left[\frac{dy}{dx}\right]_Q = -\frac{a}{h}$$

$$\Rightarrow \quad \left[\frac{dy}{dx}\right]_P = \left[\frac{dy}{dx}\right]_Q$$

Hence, the tangents are parallel.

52. We have,

$$f(x) = xe^{-x}$$

On differentiating both sides w.r.t.   , we get

$$f'(x) = e^{-x}(1-x)$$

Put $\quad f'(x) = 0$

$$\Rightarrow \quad x = 1$$

Now, $\quad f(0) = 0$

and $\quad \lim_{x\to\infty} f(x) = \lim_{x\to\infty} xe^{-x}$

$$= \lim_{x\to\infty}\frac{x}{e^x} = \lim_{x\to\infty}\frac{1}{e^x} = 0$$

Hence, the greatest value of f(x) is $\frac{1}{e}$.

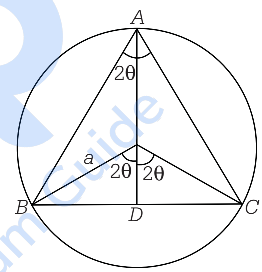
53. We have,

AD = a + a cos 2θ and BC = 2BD = 2a sin2θ

$\therefore$ Area of ΔABC is given by

$$\Delta = \frac{1}{2}BC \times AD = \frac{2}{2}a^2(\sin 2\theta + \sin 2\theta\cos 2\theta)$$

$$\Rightarrow \quad \Delta = a^2\sin 2\theta + \frac{1}{2}a^2\sin 4\theta$$



On differentiating both sides w.r.t. θ, we get

$$\frac{d\Delta}{d\theta} = 2a^2\cos 2\theta + 2a^2\cos 4\theta$$

For maximum or minimum, put $\frac{d\Delta}{d\theta} = 0$.

$$\Rightarrow \quad \cos 2\theta = -\cos 4\theta$$

$$\Rightarrow \quad 2\theta = \pi - 4\theta$$

$$\Rightarrow \quad \theta = \frac{\pi}{6}$$

Now, $\frac{d^2\Delta}{d\theta^2} = -2a^2\sin 2\theta \times 2 - 2a^2\sin 4\theta \times 4$

$$= -4a^2\sin 2\theta - 8a^2\sin 4\theta$$

At $\quad \theta = \frac{\pi}{6}, \frac{d^2\Delta}{d\theta^2} = -4a^2\sin\frac{\pi}{3} - 8a^2\sin\frac{2\pi}{3}$

$$= -4a^2 \times \frac{\sqrt{3}}{2} - 8a^2 \times \frac{\sqrt{3}}{2}$$

$$= -6a^2\sqrt{3} < 0$$

Hence, Δ is maximum for $\theta = \frac{\pi}{6}$.

54. Clearly, f(x) is defined, if

$$25 - x^2 > 0$$
$$\Rightarrow \quad -5 < x < 5$$

Let $y = \log_5(25 - x^2)$, then

$$5^y = 25 - x^2$$
$$\Rightarrow \quad x^2 = 25 - 5^y$$
$$\Rightarrow \quad x = \pm\sqrt{25 - 5^y}$$

For x to be real, we must have

$$25 - 5^y \geq 0$$
$$\Rightarrow \quad 5^y \leq 25$$
$$\Rightarrow \quad y \leq 2$$

Also, $y = f(x) \to -\infty$ as $x \to \pm 5$.

Hence, range $(f) = (-\infty, 2]$

55. We know that, the image of the point (x_1, y_1, z_1) in the plane $ax + by + cz + d = 0$ is given by

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$
$$= \frac{-2(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2}$$

So, the image of the point P (1, 3, 4) in the plane $2x - y + z + 3 = 0$ is given by

$$\frac{x-1}{2} = \frac{y-3}{-1} = \frac{z-4}{1} = \frac{-2(2-3+4+3)}{4+1+1}$$
$$\Rightarrow \quad \frac{x-1}{2} = \frac{y-3}{-1} = \frac{z-4}{1} = -2$$
$$\Rightarrow \quad x = -3, y = 5, z = 2$$

Hence, the required point is $(-3, 5, 2)$.

56. The lines are parallel to the vectors $\mathbf{b}_1 = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$ and $\mathbf{b}_2 = -\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$.

$\therefore$ The plane normal to the vector $\mathbf{n}$ is

$$\mathbf{b}_1 \times \mathbf{b}_2 = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2 & -1 \\ -1 & 1 & -2 \end{vmatrix} = -3\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$$

The required plane passes through $(\hat{\mathbf{i}} + \hat{\mathbf{j}})$ and is normal to the vector $\mathbf{n}$.

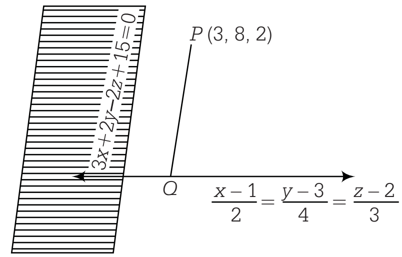
$\therefore$ Its equation is

$$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$$
$$\Rightarrow \quad \mathbf{r} \cdot (-3\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 3\hat{\mathbf{k}}) = (\hat{\mathbf{i}} + \hat{\mathbf{j}}) \cdot (-3\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 3\hat{\mathbf{k}})$$
$$\Rightarrow \quad \mathbf{r} \cdot (-3\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 3\hat{\mathbf{k}}) = -3 + 3$$
$$\Rightarrow \quad \mathbf{r} \cdot (-\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) = 0$$

57. Let P (3, 8, 2) be the given point.

Let the equation of a line through P (3, 8, 2) and parallel to the plane $3x + 2y - 2z + 15 = 0$ be

$$\frac{x-3}{a} = \frac{y-8}{b} = \frac{z-2}{c} \qquad \ldots(i)$$



Then, normal to the plane is perpendicular to the parallel line.

Then, $\quad 3a + 2b - 2c = 0 \qquad \ldots(ii)$

Line (i) intersects the line $\frac{x-1}{2} = \frac{y-3}{4} = \frac{z-2}{3}$ at .

$$\therefore \quad \begin{vmatrix} 3-1 & 8-3 & 2-2 \\ a & b & c \\ 2 & 4 & 3 \end{vmatrix} = 0$$
$$\Rightarrow \quad \begin{vmatrix} 2 & 5 & 0 \\ a & b & c \\ 2 & 4 & 3 \end{vmatrix} = 0$$
$$\Rightarrow \quad 2(3b - 4c) - 5(3a - 2c) = 0$$
$$\Rightarrow \quad 15a - 6b - 2c = 0 \qquad \ldots(iii)$$

On solving Eqs. (ii) and (iii) by cross-multiplication, we get

$$\frac{a}{-4-12} = \frac{b}{-30+6} = \frac{c}{-18-30}$$
$$\Rightarrow \quad \frac{a}{-16} = \frac{b}{-24} = \frac{c}{-48} \Rightarrow \frac{a}{2} = \frac{b}{3} = \frac{c}{6}$$

Substituting a, b and c in Eq. (i), we get

$$\frac{x-3}{2} = \frac{y-8}{3} = \frac{z-2}{6} = \lambda \qquad \text{[say]}$$

Let the coordinates of Q be $(2\lambda + 3, 3\lambda + 8, 6\lambda + 2)$.

It lies on the line $\frac{x-1}{2} = \frac{y-3}{4} = \frac{z-2}{3}$

$$\therefore \quad \lambda + 1 = \frac{3\lambda + 5}{4} = 2\lambda$$
$$\Rightarrow \quad \lambda = 1$$

$\therefore$ Coordinate of $Q = (2 + 3, 3 + 8, 6 + 2)$ i.e. $Q = (5, 11, 8)$.

Hence, $PQ = \sqrt{(5-3)^2 + (11-8)^2 + (8-2)^2}$

$$= \sqrt{4 + 9 + 36} = 7$$

58. By total probability theorem,

$$\text{Required probability} = \frac{1}{2}\times\left(\frac{3}{5}\times 1+\frac{2}{5}\times\frac{1}{2}\right)$$
$$+\frac{1}{2}\left(\frac{{}^3C_2}{{}^5C_2}\times 1+\frac{{}^2C_2}{{}^5C_2}\times\frac{1}{3}+\frac{{}^3C_1\times{}^2C_1}{{}^5C_2}\times\frac{2}{3}\right)$$
$$=\frac{1}{2}\left(\frac{3}{5}+\frac{1}{5}\right)+\frac{1}{2}\left(\frac{3}{10}+\frac{1}{30}+\frac{2}{5}\right)$$
$$=\frac{2}{5}+\frac{1}{2}\left(\frac{9+1+12}{30}\right)$$
$$=\frac{2}{3}+\frac{22}{60}=\frac{24+22}{60}$$
$$=\frac{46}{60}=\frac{23}{30}$$

59. We have,

$$\sum_{k=1}^{16} D_k = \begin{vmatrix} a & \left(\sum_{k=1}^{16} 2^k\right) & 2^{16}-1 \\ b & 3\left(\sum_{k=1}^{16} 4^4\right) & 2(4^{16}-1) \\ c & 7\left(\sum_{k=1}^{16} 8^k\right) & 4(8^{16}-1)\end{vmatrix}$$

$$\Rightarrow \sum_{k=1}^{16} D_k = \begin{vmatrix} a & 2\left(\frac{2^{16}-1}{2-1}\right) & 2^{16}-1 \\ b & 3\times 4\left(\frac{4^{16}-1}{4-1}\right) & 2(4^{16}-1) \\ c & 7\times 8\left(\frac{8^{16}-1}{8-1}\right) & 4(8^{16}-1)\end{vmatrix}$$

$$\Rightarrow \sum_{k=1}^{16} D_k = \begin{vmatrix} a & 2(2^{16}-1) & (2^{16}-1) \\ b & 4(4^{16}-1) & 2(4^{16}-1) \\ c & 8(8^{16}-1) & 4(8^{16}-1)\end{vmatrix}$$

$$\Rightarrow \sum_{k=1}^{16} D_k = 2\begin{vmatrix} a & (2^{16}-1) & (2^{16}-1) \\ b & 2(4^{16}-1) & 2(4^{16}-1) \\ c & 4(8^{16}-1) & 4(8^{16}-1)\end{vmatrix}$$

[taking 2 common from C_2]

$$\Rightarrow \sum_{k=1}^{16} D_k = 2\times 0 = 0$$

60. We have,

$$x^2-4x+4y^2=12$$
$$\Rightarrow (x-2)^2+4(y-0)^2=8$$
$$\Rightarrow \frac{(x-2)^2}{(2\sqrt{2})^2}+\frac{(y-0)^2}{(\sqrt{2})}=1$$

which is an ellipse whose major and minor axes are $2a = 2\sqrt{2}$ and $2b = \sqrt{2}$ repectively.

$\therefore$ Its eccentricity e is given by

$$e=\sqrt{1-\frac{b^2}{a^2}}$$
$$=\sqrt{1-\frac{2}{8}}=\frac{\sqrt{3}}{2}$$

61. Let $I=\int \frac{1}{\sin\left(x-\frac{\pi}{3}\right)\cos x}dx$

$$=\frac{1}{\cos\frac{\pi}{3}}\int\frac{\cos\left\{x-\left(x-\frac{\pi}{3}\right)\right\}}{\sin\left(x-\frac{\pi}{3}\right)\cos x}dx$$
$$=2\int\frac{\cos x\cos\left(x-\frac{\pi}{3}\right)+\sin x\sin\left(x-\frac{\pi}{3}\right)}{\sin\left(x-\frac{\pi}{3}\right)\cos x}dx$$
$$=2\int\left\{\cot\left(x-\frac{\pi}{3}\right)+\tan x\right\}dx$$
$$=2\left\{\log\left|\sin\left(x-\frac{\pi}{3}\right)\right|+\log|\sec x|\right\}+C$$
$$=2\log\left|\sin\left(x-\frac{\pi}{3}\right)\sec x\right|+C$$

62. The equation of any tangent to $y^2=8ax$ is

$$y=mx+\frac{2a}{m} \quad \ldots(i)$$

If it touches $x^2+y^2=2a^2$, then

$$\left(\frac{2a}{m}\right)^2=2a^2(1+m^2)$$

$[\because c^2=a^2(1+m^2)]$

$$\Rightarrow 2=m^2(m^2+1)$$
$$\Rightarrow m^4+m^2-2=0$$
$$\Rightarrow (m^2+2)(m^2-1)=0$$
$$\Rightarrow m^2-1=0$$
$$\Rightarrow m=\pm 1$$

Putting the values of m in Eq. (i), we get $y=\pm(x+2a)$ as the equations of common tangents.

63. The equation of the plane through the intersection of the planes $x + y + z = 1$ and $2x + 3y - z + 4 = 0$ is

$$(x + y + z - 1) + \lambda (2x + 3y - z + 4) = 0$$

or $(2\lambda + 1) x + (3\lambda + 1) y + (1 - \lambda) z + 4\lambda - 1 = 0$ ...(i)

It is parallel to X-axis.

i.e. $\dfrac{x}{1} = \dfrac{y}{0} = \dfrac{z}{0}$

$\therefore\ 1 (2\lambda + 1) + 0 \times (3\lambda + 1) + 0 (1 - \lambda) = 0$

$\Rightarrow\ \lambda = -\dfrac{1}{2}$

On substituting $\lambda = -\dfrac{1}{2}$ in Eq. (i), we get

$$\left(2 \times -\frac{1}{2} + 1\right)x + \left(3 \times \frac{-1}{2} + 1\right)y + \left(1 + \frac{1}{2}\right) z + 4 \times \left(-\frac{1}{2}\right) - 1 = 0$$

$\Rightarrow\ \dfrac{-1}{2}y + \dfrac{3}{2}z - 3 = 0$

$\Rightarrow\ y - 3z + 6 = 0$

which is the equation of the required plane.

64. We have, $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -2 \\ 1 & 1 & 0 \end{vmatrix} = 2\hat{i} - 2\hat{j} + \hat{k}$

and $|\mathbf{a} \times \mathbf{b}| = \sqrt{2^2 + 2^2 + 1^2} = \sqrt{4 + 4 + 1} = 3$

$\therefore\ |(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}| = |\mathbf{a} \times \mathbf{b}|\,|\mathbf{c}| \sin 30^\circ$

$= \dfrac{3}{2}|\mathbf{c}|$

Given, $|\mathbf{c} - \mathbf{a}| = 2\sqrt{2}$

$\Rightarrow\ |\mathbf{c} - \mathbf{a}|^2 = 8$

$\Rightarrow\ |\mathbf{c}|^2 + |\mathbf{a}|^2 - 2 (\mathbf{a} \cdot \mathbf{c}) = 8$

$\Rightarrow\ |\mathbf{c}|^2 + 9 - 2|\mathbf{c}| = 8$

$\Rightarrow\ |\mathbf{c}|^2 - 2|\mathbf{c}| + 1 = 0$

$\Rightarrow\ |\mathbf{c}| = 1$

Hence, $|(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}| = \dfrac{3}{2}$

65. The given differential equation can be written as

$$\left(\frac{d^2y}{dx^2}\right)^4 = \left\{y + \left(\frac{dy}{dx}\right)^2\right\}$$

Clearly, its order and degree are 2 and 4 respectively.

66. Let h be the height, r be the radius of the base and be the volume of the cone at time t. Then,

$$V = \frac{1}{3}\pi r^2 h$$

$\Rightarrow\ V = \dfrac{2}{3}\pi r^3$ $[\because h = 2r, \text{ given}]$

On differentiating both sides w.r.t. , we get

$$\frac{dV}{dt} = 2\pi r^2 \frac{dr}{dt}$$

$\Rightarrow\ 40 = 2 (10)^4 \dfrac{dr}{dt}$

$\left[\because \pi r^2 = 1\ m^2 = 10^4\ cm^2 \text{ and } \dfrac{dV}{dt} = 40\ cm^3/s, \text{ given}\right]$

$\Rightarrow\ \dfrac{dr}{dt} = \dfrac{2}{1000}$ cm/s $= 0.002$ cm/s

67. Let S be the sum of the given series.

i.e. $S = 1 + \dfrac{2}{3} + \dfrac{6}{3^2} + \dfrac{10}{3^3} + \ldots$

$\Rightarrow\ (S - 1) = \dfrac{2}{3} + \dfrac{6}{3^2} + \dfrac{10}{3^3} + \ldots$ ...(i)

$\Rightarrow\ (S - 1) \times \dfrac{1}{3} = \dfrac{2}{3^2} + \dfrac{6}{3^3} + \dfrac{10}{3^4} + \ldots$ ...(ii)

On subtracting Eq. (ii) from Eq. (i), we get

$$\frac{2}{3}(S - 1) = \frac{2}{3} + \frac{4}{3^2} + \frac{4}{3^3} + \frac{4}{3^4} + \ldots$$

$\Rightarrow\ \dfrac{2}{3}(S - 1) = \dfrac{2}{3} + \dfrac{\frac{4}{3^2}}{1 - \frac{1}{3}}$

$\Rightarrow\ \dfrac{2}{3}(S - 1) = \dfrac{2}{3} + \dfrac{2}{3}$

$\Rightarrow\ S - 1 = 2$

$\Rightarrow\ S = 3$

68. Let the equation of the line be

$\dfrac{x}{a} + \dfrac{y}{b} = 1$ ...(i)

which passes through (4, 3).

$\therefore\ \dfrac{4}{a} + \dfrac{3}{b} = 1$ ...(ii)

It is given that $a + b = -1$ ...(iii)

On solving Eqs. (i) and (ii), we get

$a = -2, b = 1$ or $a = 2, b = -3$

On substituting the values of a and b in Eq. (i), we get

$$\frac{x}{-2} + \frac{y}{1} = 1 \text{ and } \frac{x}{2} + \frac{y}{-3} = 1$$

69. Given points will be collinear, if

$$\begin{vmatrix} a & b & 1 \\ b & a & 1 \\ a^2 & -b^2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} a & b & 1 \\ b-a & a-b & 0 \\ a^2-a & -b^2-b & 0 \end{vmatrix} = 0$$

[applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$]

$$\Rightarrow \quad (a-b)\begin{vmatrix} a & b & 1 \\ -1 & 1 & 0 \\ a^2-a & -b^2-b & 0 \end{vmatrix} = 0$$

$$\Rightarrow \quad (a-b)(b^2+b-a^2+a) = 0$$

$$\Rightarrow \quad (a-b)\{(a+b)-(a^2-b^2)\} = 0$$

$$\Rightarrow \quad (a-b)(a+b)(1-a+b) = 0$$

$$\Rightarrow \quad a = b \text{ or } a+b = 0 \text{ or } a = 1+b$$

70. We have,

$$x^2+9 < (x+3)^2 < 8x+25$$

$$\Rightarrow \quad x^2+9 < x^2+6x+9 < 8x+25$$

$$\Rightarrow \quad x^2+9 < x^2+6x+9$$

$$\text{and} \quad x^2+6x+9 < 8x+25$$

$$\Rightarrow \quad 6x > 0 \text{ and } x^2-2x-16 < 0$$

$$\Rightarrow \quad x > 0 \text{ and } 1-\sqrt{17} < x < 1+\sqrt{17}$$

$$\Rightarrow \quad 0 < x < 1+\sqrt{17}$$

$$\therefore \quad x = 1, 2, 3, 4, 5$$