Indian Forest Services (Main)
Examination-2025

DJSM-U-MATH

MATHEMATICS

Paper – I

*Time Allowed : **Three Hours***

*Maximum Marks : **200***

Question Paper Specific Instructions

Please read each of the following instructions carefully before attempting questions :

*There are **EIGHT** questions in all, out of which **FIVE** are to be attempted.*

*Questions no. **1** and **5** are **compulsory**. Out of the remaining **SIX** questions, **THREE** are to be attempted selecting at least **ONE** question from each of the two Sections **A** and **B**.*

Attempts of questions shall be counted in sequential order. Unless struck off, attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question-cum-Answer Booklet must be clearly struck off.

All questions carry equal marks. The number of marks carried by a question/part is indicated against it.

*Answers must be written in **ENGLISH** only.*

Unless otherwise mentioned, symbols and notations have their usual standard meanings.

Assume suitable data, if necessary, and indicate the same clearly.

SECTION A

Q1. (a) If a subspace W of $\mathbb{R}^4$ is generated by the vectors (3, 8, – 3, – 5), (1, – 2, 5, – 3) and (2, 3, 1, – 4), then find a basis and dimension of W. Extend that basis to get a basis of $\mathbb{R}^4$. *8*

(b) Find a row echelon matrix which is row equivalent to

$$A = \begin{bmatrix} 0 & 0 & -2 & 3 & 1 \\ 2 & 4 & 1 & 4 & 3 \\ 1 & 2 & -3 & 1 & 2 \\ 4 & 8 & 2 & 3 & 5 \end{bmatrix}$$

and find the rank of A. *8*

(c) Amreek has n number of children by his first wife. Shaina has (n + 1) children by her first husband. They marry and have children of their own also. The whole family now has 12 children. It is assumed that children of Amreek from his first wife do not fight among themselves, and likewise, children of Shaina by her first husband do not fight among themselves. Find the maximum possible number of fights between children that can take place. *8*

(d) Prove that : *8*

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{\sqrt{\pi}\,\Gamma(2n+1)}{2^{2n}\,\Gamma(n+1)}$$

(e) Find the equation of the plane passing through the points (2, 2, 1) and (9, 3, 6) and perpendicular to the plane 2x + 6y + 6z = 9. *8*

Q2. (a) For the companion matrix C

$$\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix}$$

of a n^{th} degree polynomial

$$\phi(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0,$$

prove that

(i) the characteristic polynomial is $\phi(\lambda)$.

(ii) if λ_i is an eigenvalue of C, then $x_i = [1 \;\; \lambda_i \;\; \lambda_i^2 \; \dots \; \lambda_i^{n-1}]^T$ is the associated eigenvector.

(iii) if $\lambda_1, \lambda_2, \dots, \lambda_n$ are distinct eigenvalues of C, then $V^{-1} C V = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$,

where $V = \begin{bmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_n \\ \vdots & & & \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \dots & \lambda_n^{n-1} \end{bmatrix}$. 10

(b) (i) Find the expansion of $(\sin^{-1} x)^2$ in terms of ascending powers of x. 8

(ii) Evaluate : 7

$$\lim_{x \to \infty} (x\sqrt{x^2+1} - x^2)$$

(c) (i) Find the shortest distance between the straight lines

$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-5}{4} = \frac{y-4}{4} = \frac{z-5}{5}$.

Also show that the lines are coplanar. 8

(ii) Show that the enveloping cylinder of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, whose generators are parallel to the line $\frac{x}{0} = \frac{y}{\pm\sqrt{a^2 - b^2}} = \frac{z}{c}$, meets the plane z = 0 in circles. 7

Q3. (a) Prove that $\frac{y-x}{1+y^2} < \tan^{-1} y - \tan^{-1} x < \frac{y-x}{1+x^2}$, $0 < x < y$.

Hence or otherwise, show that $\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\left(\frac{4}{3}\right) < \frac{\pi}{4} + \frac{1}{6}$. *10*

(b) (i) Diagonalize the quadratic form

$5x_1^2 + 26x_2^2 + 10x_3^2 + 4x_2x_3 + 14x_3x_1 + 6x_1x_2$.

Show that it is positive semi-definite and find a non-zero set of values of x_1, x_2, x_3 which makes the diagonalized form zero. *8*

(ii) If W is a subspace of a finite dimensional vector space V(F), then prove that W is finite dimensional and $\dim W \leq \dim V$. Also, prove that $\dim W = \dim V$ if and only if $W = V$. *7*

(c) Find the equation of the cone whose vertex is the point (1, 2, 3) and guiding curve is : *15*

$$x^2 + y^2 + z^2 = 4, \quad x + y + z = 1.$$

Q4. (a) Find the equations of the spheres passing through the circle

$$x^2 + y^2 + z^2 - 5 = 0, \quad 2x + 3y + z - 3 = 0$$

and touching the plane $3x + 4z - 15 = 0$. *10*

(b) (i) For the linear operator $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T(x, y, z) = (x + y + z,\ 2y + z,\ 2y + 3z),$$

find the eigenvalues and the basis for eigenspace. *8*

(ii) Prove that the necessary and sufficient condition for a linear transformation $y = Ax$ to preserve lengths is that the matrix A is orthogonal. *7*

(c) Prove that the area included between the folium

$$x^3 + y^3 = 3axy$$

and its asymptote is equal to the area of its loop. *15*

SECTION B

Q5. (a) Solve the differential equation

$$\frac{dy}{dx} + \frac{y}{(1-x^2)^{3/2}} = \frac{x + (1-x^2)^{1/2}}{(1-x^2)^2}.$$ *8*

(b) Solve the differential equation

$$x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = \cos(\log x) + x \sin(\log x).$$ *8*

(c) The displacement of a particle in a straight line is given by the equation $x = a \cos nt + b \sin nt$. Show that the particle describes a simple harmonic motion whose amplitude is $\sqrt{a^2 + b^2}$ and period is $\frac{2\pi}{n}$. *8*

(d) A light ladder is supported on a rough floor and leans against a smooth wall. How far up the ladder can a man climb without slipping taking place ? *8*

(e) For vector fields $\vec{u}$ and $\vec{v}$, prove that

$$\vec{\nabla} \times (\vec{u} \times \vec{v}) = \vec{u}(\vec{\nabla} \cdot \vec{v}) - \vec{v}(\vec{\nabla} \cdot \vec{u}) + (\vec{v} \cdot \vec{\nabla})\vec{u} - (\vec{u} \cdot \vec{\nabla})\vec{v}.$$ *8*

Q6. (a) Given that $y = x + \frac{1}{x}$ is a solution of the differential equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0.$$

Find the other linearly independent solution and write down the general solution of the given differential equation. *10*

(b) A projectile is launched with initial speed u making an angle $(90 - \theta)$ with the vertical. An air resistance force $-\beta\vec{v}$ $(\beta > 0)$ acts upon the projectile, where $\vec{v}$ is the instantaneous velocity. Find the velocity $\vec{v}$, and show that the position vector $\vec{r}$ at any time t is

$$\vec{r} = \frac{mu}{\beta}(\cos\theta\,\hat{j} + \sin\theta\,\hat{k})\left(1 - e^{-\frac{\beta t}{m}}\right) - \frac{mg}{\beta}\left(t + \frac{m}{\beta}e^{-\frac{\beta t}{m}} - \frac{m}{\beta}\right)\hat{k},$$

where m is the mass of the projectile. *15*

(c) (i) Find the directional derivative of

$$F(x, y, z) = xy^2 - 4x^2y + z^2$$

at $(1, -1, 2)$ in the direction of $6\hat{i} + 2\hat{j} + 3\hat{k}$. Also find its maximum value. *5*

(ii) Verify Stokes' theorem for $\vec{f} = (x + y)\hat{i} + yz^2\hat{j} + y^2z\hat{k}$, where S is the upper surface of the sphere $x^2 + y^2 + z^2 = 1$ over $z = 0$ and Γ is its boundary in the xy plane. *10*

Q7. (a) A framework ABCD consists of four equal, light rods smoothly jointed together to form a square. It is suspended from a peg at A, and a weight w is attached to C, the framework being kept in shape by a light rod connecting B and D. Find the thrust in this rod. *10*

(b) Use the method of variation of parameters to show that the solution of

$$\frac{d^2y}{dx^2} + k^2y = \phi(x),$$

satisfying the conditions $y(0) = 0 = \left.\frac{dy}{dx}\right|_{x=0}$ is given by

$$y(x) = \frac{1}{k}\int_0^x \phi(t)\sin k(x - t)\,dt.$$ *15*

(c) (i) Determine $\vec{a}$ such that

$$\frac{d\hat{T}}{ds} = \vec{a} \times \hat{T}, \quad \frac{d\hat{N}}{ds} = \vec{a} \times \hat{N}, \quad \frac{d\hat{B}}{ds} = \vec{a} \times \hat{B}$$

represents Frenet-Serret formulae. *5*

(ii) A curve in a space is represented by

$$\vec{r} = e^t \cos t\,\hat{i} + e^t \sin t\,\hat{j} + e^t\hat{k}.$$

Find the curvature and principal normal of this curve at $t = 0$. *10*

Q8. (a) (i) Find the values of α and β such that the vectors

$$\vec{f} = (\alpha x + y)\,\hat{i} + (y - 3z)\,\hat{j} + (x + \alpha z)\,\hat{k} \text{ and } \vec{g} = \frac{x\,\hat{i} + y\,\hat{j}}{x^2 + y^2} + \hat{k}$$

are solenoidal. *3*

(ii) Show that the vector

$$\vec{f} = (2x - yz)\,\hat{i} + (2y - zx)\,\hat{j} + (2z - xy)\,\hat{k}$$

is irrotational and find a scalar function ϕ such that $\vec{f} = \text{grad}\,\phi$. *7*

(b) Find the general and singular solutions of

$$3xy = 2px^2 - 2p^2, \quad p = \frac{dy}{dx}.$$ *15*

(c) A solid hemisphere is placed with its base inclined to the surface of a liquid, in which it is completely immersed, at a given angle α. Show that if the resultant thrust on the curved portion of the surface is equal to twice the weight of the liquid displaced, then $\tan \alpha = 2$. *15*